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$N = 1$ Heterotic/F-Theory Duality

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Abstract

We review aspects of $N = 1$ duality between the heterotic string and F-theory. After a description of string duality intended for the non-specialist the framework and the constraints for heterotic/F-theory compactifications are presented. The computations of the necessary Calabi-Yau manifold and vector bundle data, involving characteristic classes and bundle moduli, are given in detail. The matching of the spectrum of chiral multiplets and of the number of heterotic five-branes respectively F-theory three-branes, needed for anomaly cancellation in four-dimensional vacua, is pointed out. Several examples of four-dimensional dual pairs are constructed where on both sides the geometry of the involved manifolds relies on del Pezzo surfaces.

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1 Introduction

The standard model of elementary particle physics has been verified to high precision. All fermions in the standard model have been discovered, only the Higgs particle is missing, discovering it, one is tempted to say that 'particle physics is finished'.

However, there are a few conceptual difficulties with the standard model, which indicate that one should expect new physics beyond.

For example, one would like to know, why there are nineteen arbitrary parameters in the standard model and how they can be reduced. Also, one should be able to explain the origin of the three fermion generations and how their masses can be predicted.

Another striking question is the gauge hierarchy problem. It refers to the fact that the weak mass scale, set by the Higgs mechanism at $m_{weak} = 250$ GeV, is not stable under radiativ corrections. If gravity is included the hierarchy problem demands for explanation of the relativ smallness of the weak scale, compared with the Planck scale ($m_{Pl} = 10^{19}$ GeV).

Further important questions are the following: why is the standard model gauge group $SU(3) \times SU(2) \times U(1)$; why does the cosmological constant vanish; why our world is four-dimensional and how gravity is included at the quantum level?

A first step in answering these questions was the introduction of grand unified theories (GUTs). They are based on gauge symmetry groups like $SU(5)$ or $SO(10)$. These groups are expected to appear at high energies, thus should unify the gauge couplings, and expected to be broken at low energies to the standard model gauge group. Further, they explain charge quantization and the relationship between quarks and leptons. But, they do not provide answers on the hierarchy problem and other related questions. In particular, they predict the proton decay, which has been not observed in nature up to now.

A next step was the introduction of global supersymmetry (SUSY) which essentially solves the hierarchy problem. Due to the existence of superpartners for all particles, the quadratic self-mass divergences of the scalar Higgs field vanish, and therefore the weak scale is stabilized. So far no supersymmetric partners have been discovered, so supersymmetry, if it exists, has to be broken. Now, the spontaneous breaking of global supersymmetry leads to the existence of massless fermionic states, the Goldstinos, which are also not observed.

As an alternative one can introduce local supersymmetry, which in contrast to global SUSY incorporates gravity (SUGRA), avoids the Goldstinos, makes the vanishing of

the cosmological constant possible, but is like gravity itself non-renormalizable.

The most promising candidat for solving all these problems is the superstring theory [1],[2]. A string is an extended object, with a size of $\sim 10^{-33}cm$, and replaces the point particles as elementary objects. More precisely, all of the known elementary particles and gauge bosons can be realized as the different excitation modes of a string. Among the excitations one finds a spin two massless excitation, which can be identified as the graviton, and moreover, supergravity is incorporated as low energy effective theory. Another nice property of string theory is, due to the cancellation of anomalies, the dimension of space-time and gauge groups are fixed. The cancellation of the conformal anomaly fixes the space-time dimension to be 10 for the superstrings and 26 for the bosonic strings. The cancellation of gauge and gravitational anomalies fixes the gauge group to be either $E_8 \times E_8$ or $SO(32)$ in ten dimensions. In particular, there are only five consistent superstring theories in ten dimensions. The type IIA, the type IIB string, the type I with $SO(32)$ gauge group and two heterotic string theories with $E_8 \times E_8$ resp. $SO(32)$ gauge group. The common features of all five string theories are: they are space-time supersymmetric and contain a graviton, an antisymmetric tensor field and the so called dilaton in their massless spectrum. Moreover, the vacuum expectation value of the dilaton controlls (at the tree level) the gauge couplings. More precisely, the bosonic spectrum of the type I and type II theories appears in two sectors, the Ramond-Ramond (RR) or Neveu-Schwarz-Neveu-Schwarz (NS-NS) sector, depending on the boundary conditions of the worldsheet fermions, which are periodic (RR) resp. anti-periodic (NS-NS). In particular, the type IIA and IIB string theories contain only closed strings and have two gravitinos, with opposite chiralty in IIA and the same chirality IIB, whereas both lead to $N = 2$ SUGRA in their low energy limit. In contrast, type I contains, in addition to closed also open strings, has one gravitino and leads to $N = 1$ SUGRA in the field theory limit. The two heterotic string theories contain closed strings, they have one gravitino and lead also to $N = 1$ SUGRA in the field theory limit.

Although strings are the most promising candidates for grand unified model building, there are a few inherent stringy problems. First of all, since the critical dimension for consistent string theories is ten, one has to compactify six of them. This means that a vacuum configuration must be specified, in order to make contact with the four-dimensional world. Second, if string theory claims to be a unique fundamental theory, why there are five consistent theories in ten dimensions? Third, string theory incorporates not only the right elementary excitations and appropriate gauge groups for

successful GUT predictions, they also contain gravity and thus should make predictions for the gravitational coupling strength, i.e. for Newton's constant, predictions which fail in perturbative heterotic string theory. Not only the latter fact, but also a desirable understanding of supersymmetry breaking leads to a fourth point, a nonperturbative formulation of string theory, including nonperturbative effects at strong coupling. The necessity of a nonperturbative formulation is also supported by the fact, that the specification of the vacuum configuration can be understood dynamically.

Something one can do, at least, is to combine some demands as: maintaining conformal invariance, unbroken supersymmetry and vanishing of the gauge anomaly (if ten-dimensional gauge groups are involved) in order to restrict possible vacuum configurations. It has been shown that this favors vacuum configurations based on Calabi-Yau compactifications, so that one ends up with vacua which are products of the four-dimensional Minkowski space M_4 and a complex three-dimensional, Ricci-flat Kähler manifold X , i.e. X has vanishing first Chern-class, and so preserving conformal invariance. Imposing the condition of Ricci-flatness to Kähler manifolds of complex dimension n , reduces the holonomy group $U(n)$ to $SU(n)$, which is stated by a theorem of Calabi and Yau.

In fact, compactifications, say, of the heterotic string on a Calabi-Yau threefolds leads to vacua with $N = 1$ supersymmetry in four dimensions. The gauge anomaly is cancelled by embedding the holonomy group in the gauge group $E_8 \times E_8$, which is called 'the standard embedding' (i.e. the $SU(3)$ spin connection of X is identified with an $SU(3)$ subgroup of E_8), thus leaving an 'hidden' E_8 and an E_6 as viable GUT group.

Now, why are there five consistent string theories in ten dimensions? This question is expected to be answered due to the establishment of various types of duality symmetries between different string vacua, which were inspired by duality considerations in supersymmetric field theories. More precisely, Montonen and Olive stated that the $N = 4$ supersymmetric Yang-Mills theory admits a weak-strong coupling duality, exchanging elementary (electric charged) with solitonic (magnetic charged) states. In string theory, a similar duality was observed in heterotic string compactifications on a six-dimensional torus, which leads to $N = 4$ supersymmetry in four dimensions. This duality, called S -duality [3], is also a strong-weak coupling duality and so a nonperturbative symmetry of this theory. Also, the uncompactified type IIB superstring possesses such a symmetry, transforming states (BPS p-branes) from the Neveu-Schwarz sector into states (Dirichlet p-branes) of the Ramond sector.

All these dualities are only symmetries of one string theory, however, it has been shown that there are also duality symmetries relating the weak coupling region of one string theory to the strong coupling region of another and vice versa. The duality is called string-string duality. For example, the type I theory is conjectured to be dual to the $SO(32)$ theory in ten dimensions [4], [5]. Another example is the heterotic string compactified on T^4 which is conjectured to be dual to the type IIA string on $K3$ in six dimensions [6], [7]. Kachru and Vafa [8] stated the duality between the heterotic string compactified on $K3 \times T^2$ and the type IIA string on Calabi-Yau threefold which leads to $N = 2$ supersymmetry in four dimensions.

Tests of these dualities would certainly involve computations at strong coupling, but since string theory is only formulated perturbatively, this seems to be offhand impossible. Fortunately supersymmetry leads to certain non-renormalization theorems and therefore some weak coupling calculations are exact, they do not require nonperturbative corrections. So one has to look for certain quantities which are uncorrected. In theories with extended supersymmetry, such quantities are related to BPS states. BPS states saturate the Bogomolny bound between mass and charge; their mass is completely determined in terms of their charge. For example, they are relevant in tests of the nonperturbative $SL(2, \mathbf{Z})$ duality of the $N = 4$ theory in four dimensions, obtained by compactification of the heterotic string on T^6 [9], and in the study of one-loop threshold corrections to gauge and gravitational couplings in $N = 2$ heterotic string compactifications [10].

In particular, tests could successfully performed in a broad class of type II compactifications on Calabi-Yau threefolds and heterotic vacua on $K3 \times T^2$ with $N = 2$ supersymmetry in four dimensions (for a review c.f. [11]). Further it has been shown that using these $N = 2$ string duality symmetries, results about nonperturbative effects in string theory and field theory can be derived, for example, the computation of the nonperturbative heterotic $N = 2$ prepotential or nonperturbative effects in $N = 2$ field theory like Seiberg/Witten [12], [13], [14].

It has also been shown that solitonic objects, which are given in string theory by membranes and p-branes, have to be included in order to successfully test string-string duality [6], [4].

Due to the duality descriptions between different string theories, it is now believed that all five string theories are perturbative expansions of one fundamental underlying theory, called M-theory. It is not clear up to now what M-theory is, but at low energies it should reduce to eleven-dimensional supergravity. The supporting arguments which

led to the existence of M-theory are twofold.

First, the type IIA string has $N = 2$ supergravity as ten-dimensional field theory limit. It has been shown [15] that $N = 2$ supergravity in ten dimensions can be obtained by compactification of eleven-dimensional supergravity on S^1 where the radius of S^1 is related to the type IIA string coupling as $R_{11} \sim g_{IIA}^{\frac{2}{3}}$. Witten has argued [4] that in the strong coupling limit (i.e. $R_{11} \rightarrow \infty$) the type IIA string theory approaches an eleven-dimensional (Lorentz invariant) theory whose low energy limit is eleven-dimensional supergravity.

Second, it has been shown [16],[17] that the strong coupling limit of the heterotic string can be described by an eleven-dimensional theory compactified on S^1/\mathbf{Z}_2 where the $E_8 \times E_8$ gauge fields live on the two ten-dimensional boundaries respectively. As a consequence of the strong coupling scenarion, the predictions for Newton's constant are relatively close to the actual value.

As a next step one can try to derive the type IIB and $SO(32)$ resp. type I string theory from M-theory and therefore try to explain the geometric origin of the $SL(2, \mathbf{Z})$ symmetry of type IIB theory. Specifically, one can argue that type IIB compactified on S^1 has $SL(2, \mathbf{Z})$ symmetry in nine dimensions and is T-dual to type IIA theory compactified on S^1 thus M-theory on T^2 is dual to type IIB on S^1 and the $SL(2, \mathbf{Z})$ symmetry of type IIB becomes the U-duality for type IIA on S^1 [18]. The $SL(2, \mathbf{Z})$ symmetry is just the group of global diffeomorphisms of T^2 [19], [20]. But in going to ten dimensions, one has to take the zero size limit of T^2 and therefore loses the geometric description of M-theory (similarly, one can show for $SO(32)$ resp. type I theory).

Now, taking $SL(2, \mathbf{Z})$ symmetry in ten dimensions leads Vafa [21] to the introduction of a twelve-dimensional theory, called F-theory, which compactified on T^2 leads to type IIB in ten dimensions (or compactified on T^2/\mathbf{Z}_2 leads to $SO(32)$ theory whose strong coupling limit is described by a weakly coupled ten-dimensional type I theory [4], [5]). In particular, the $SL(2, \mathbf{Z})$ symmetry of the dilaton-axion field τ (a complex field constructed with the R-R axion $\tilde{\phi}$ and the NS-NS dilaton ϕ) can be interpreted as the modular invariance of T^2 and so the $SL(2, \mathbf{Z})$ symmetry of type IIB theory is geometrized.

In standard perturbative compactifications of type IIB string theory the dilaton-axion field τ is constant on the internal manifold B . As it has been shown [21], [22], [23], compactifications of F-theory to lower dimensions can be formulated as type IIB theory on B with varying τ over B . Regarding F-theory in this way, avoids the difficulties

in formulating a consistent twelve-dimensional theory which would have a metric of $(10,2)$ signature. The fact that τ is identified with the complex structure modulus of T^2 and varies over B , makes one consider elliptically fibered manifolds, thus preserving the $SL(2, \mathbf{Z})$ symmetry. In a generic F-theory compactification the elliptic fiber can degenerate over a codimension one subspace Δ in the base and therefore non-trivial closed cycles in B can induce non-trivial $SL(2, \mathbf{Z})$ transformation on the fiber. Thus the dilaton is not constant, it can jump by an $SL(2, \mathbf{Z})$ transformation, a fact which has no explanation in perturbative string theory. Moreover, it has been shown that the non-trivial monodromy around closed cycles signals the presence of magnetically charged D-sevenbranes at Δ , filling the uncompactified space-time [21].

So, besides the explanation of the $SL(2, \mathbf{Z})$ symmetry of the ten-dimensional type IIB theory, F-theory provides new nonperturbative type IIB vacua on D-manifolds with varying coupling over the internal space.

Moreover, F-theory also provides an alternative description of heterotic string compactification in purely geometrical terms. In particular, it has been shown [21], [22], [23] that F-theory compactified to eight dimensions on $K3$ is dual to the heterotic string on T^2 . Once the eight-dimensional duality is established, one can extend adiabatically [24] over a base manifold B and thus obtain lower dimensional dualities. This has been done for one-dimensional B leading to $N = 1$ heterotic/F-theory dual pairs in six dimensions. Of even greater phenomenological interest is the investigation of string duality symmetries in four-dimensional string vacua with $N = 1$ space-time supersymmetry. Therefore one compactifies the heterotic string on an elliptically fibered Calabi-Yau threefold with a vector bundle, which breaks part of the $E_8 \times E_8$ group to some possibly viable GUT group. This is conjectured to be dual to F-theory on elliptically fibered Calabi-Yau fourfold. The unbroken heterotic gauge group is expected to be localized on a locus of degenerated elliptic fibers in the fourfold. Furthermore, it has been shown that consistent F-theory compactifications on Calabi-Yau fourfold require a number of space-time filling threebranes which are localized at points in the base B of the elliptic fourfold (thus, their worldvolume is a four-dimensional submanifold of space-time and given by $\mathbf{R}^4 \times p_i$, where p_i are the points in B). The number of such threebranes was determined in [25] by observing that the SUGRA equations have a solution only for a precise number of such threebranes (more correctly, this number has been shown to be proportional to the Euler characteristic of the fourfold). Under duality these threebranes should turn into heterotic fivebranes [26].

Now, if the map between the heterotic string description and F-theory can be made

precise, important nonperturbative information, like the computation of nonperturbative $N = 1$ superpotentials [27], [28], [29], [30] and supersymmetry breaking, or the reformulation of many effects in $N = 1$ field theory [31], [32], [33], [34] are expected. In addition, the way in which transitions among $N = 1$ heterotic vacua, possibly with a different number of chiral multiplets, take place, is a very important question.

Our primary topic is the study of $N = 1$ heterotic/F-theory duality in four dimensions. In the first part we will review and develop some methods and tools needed for studying four-dimensional heterotic/F-theory duality.

Let us be more precise!

In section 2, we review the constraints for consistent $E_8 \times E_8$ heterotic string compactification and give an example of how the massless spectrum is obtained.

In section 3, we first specify the class of Calabi-Yau manifolds which we will consider in the remaining part.

In addition to the specification of the manifold, we have to specify a stable vector bundle, in order to compactify the heterotic string, this breaks part of the $E_8 \times E_8$ gauge symmetry.

Actually, there are three methods for constructing stable vector bundles over elliptic fibrations: the parabolic, the spectral cover and the construction via del Pezzo surfaces which are explained and developed in [26] and [35].

We adopt here the parabolic construction, however, we compare our results to those obtained from the spectral cover construction.

We consider a $G = SU(n)$ vector bundle, with n odd, which determines a rank n complex vector bundle V of trivial determinant.

We first review, for our purposes, some facts of the parabolic construction for V an $SU(n)$ vector bundle, then we will recall what is known for n -even which was discussed in [26] and we will extend this to the case if n is odd. In contrast to [26] we do not focus on a τ -invariant point in the moduli space of $SU(n)$ bundles. This allows us to determine the net number N_{gen} of generations of chiral fermions, i.e. ($\#$ generations $-$ $\#$ antigerations) in the observable sector of the 4D unbroken gauge group. In contrast to the bundles at the τ -invariant point, which have n even and a modulo 2 condition for η , our bundles have n odd and $\pi_*(c_2(V)) = \eta$ divisible by n . In particular it will be shown that for a certain choice of the twisting line bundle on the spectral cover both approaches agree.

In the last part of section 3, we will determine the number of bundle moduli for $SU(n)$

and E_8 vector bundles on elliptically fibered threefolds using technics (a character-valued index) computed and applied in [26] for $SU(n)$ vector bundles.

In section 4, we review the framework of F-theory, the constraints for consistent four-dimensional F-theory compactification on Calabi-Yau fourfold and recall the adiabatic principle, needed to establish the duality between the heterotic string and F-theory.

In section 5.1 and 5.2, we review some facts about eight- and six-dimensional het/F-theory duality, some examples for the six-dimensional case are presented. In particular, section 5.2 provides us with the necessary tools we will later use to construct four-dimensional heterotic/F-theory models from six-dimensional ones.

In section 5.3, we show that in dual $N = 1$ heterotic/F-theory vacua the number of heterotic fivebranes necessary for anomaly cancellation matches the number of F-theory threebranes necessary for tadpole cancellation. This extends to the general case the work of Friedman, Morgan and Witten, who treated the case of embedding a heterotic $E_8 \times E_8$ bundle, leaving no unbroken gauge group, where one has a smooth Weierstrass model on the F-theory side.

Further, we compare the spectrum of chiral multiplets obtained in four-dimensional heterotic string compactification with vector bundle V with the spectrum of chiral multiplets obtained from F-theory compactification on Calabi-Yau fourfold. We find a complete matching of both spectra.

In section 6.1, we consider F-theory on smooth fourfolds X_k^4 elliptic over $dP_k \times \mathbf{P}^1$ as an example.

In section 6.2, we study a class of Calabi-Yau threefolds elliptic over dP_k and use the index formula for the number of moduli of a general $E_8 \times E_8$ bundle and compare the spectra with those obtained from the fourfolds X_k^4 .

In section 6.3, we consider the \mathbf{Z}_2 modding of the $N = 2$ models described by F-theory on $T^2 \times X_n^3$, where X_n^3 are the well known threefold Calabi-Yau over F_n .

In section 6.4, we make the corresponding modding in the dual heterotic $N = 2$ model on $T^2 \times K3$ with instanton embedding $(12 + n, 12 - n)$.

In section 6.5, we consider the standard embedding, which corresponds to a purely perturbative compactification (without fivebranes). We show that the number of threebranes also vanishes on the F-theory side, and study the 6D analog of the complete Higgsing process. We find evidence that the 4D spectrum can be obtained from compactification of the heterotic string on $K3 \times T^2/(\sigma, -1)$.

2 $N = 1$ Heterotic Vacua

2.1 Perturbative Vacua

If we study the $E_8 \times E_8$ heterotic string in a flat Minkowskian background, then the cancellation of the conformal anomaly forces us to choose a ten-dimensional spacetime. Requiring unbroken $N = 1$ spacetime supersymmetry of the effective theory, which arose from compactification on $M_4 \times Z$, is equivalent to requiring that

$$\langle \Omega | \delta \Psi | \Omega \rangle = 0 \quad (2.1)$$

for all fundamental fermionic fields in the spacetime Lagrangian where $|\Omega\rangle$ is the vacuum state of the supersymmetric field theory and $\delta \Psi$ denotes the variation of Ψ under a supersymmetry transformation. Since the effective theory that arises from heterotic string theory is $N = 1$ supergravity coupled to super Yang-Mills theory Ψ can be a gravitino ψ_i , a gluino χ^a or a dilatino λ . The gravitino transforms under supersymmetry transformation as

$$\delta \psi_i = \frac{1}{\kappa} D_i \eta + \frac{\kappa}{32g^2\phi} (\Gamma_i^{jkl} - 9\delta_i^j \Gamma^{kl}) \eta H_{jkl} \quad (2.2)$$

where Γ_i^{jkl} are anti-symmetrized products of gamma matrices, η denotes the Grassman parameter, ϕ is the dilaton, H is the field strength for the antisymmetric tensor field B and g, κ are the gravitational and gauge coupling constants. g and κ are not independent parameters, they can be reabsorbed in the dilaton so that the only independent parameter in the theory is the VEV of ϕ .

With the ansatz $H = d\phi = 0$, $\delta \psi_i = 0$ tells us that the manifold K must admit a covariant constant spinor field η , i.e. K must be a Kähler manifold with vanishing first Chern class and a metric of $SU(3)$ holonomy, i.e. K is a Calabi-Yau manifold.

The supersymmetry transformation for the gluinos χ^a are given by

$$\delta \chi^a = -\frac{1}{4g\sqrt{\phi}} \gamma^{ij} F_{ij}^a \eta \quad (2.3)$$

and $\delta \chi^a = 0$ leads to two conditions:

$$F_{ij} = F_{\bar{i}\bar{j}} = 0 \quad (2.4)$$

tells us that the vacuum gauge field A is a holomorphic connection on a holomorphic vector bundle $V \rightarrow K$ where V must be some subbundle of the $E_8 \times E_8$ bundle. The

second condition

$$g^{i\bar{j}}F_{i\bar{j}} = 0 \quad (2.5)$$

is the so called Donaldson-Uhlenbeck-Yau equation for the connection A which has a unique solution precisely if the integrability condition $\int_K \Omega^{n-1} \wedge c_1(V) = 0$ is satisfied and V is a stable ¹ bundle or a direct sum of stable bundles. This was shown by Donaldson [36] for the case of K being complex two-dimensional and by Uhlenbeck and Yau [37] for higher dimensional cases. Since the Kähler form Ω^n of K , which is the volume form of K , has nonzero integral over K , we are restricted to cases where the first Chern class $c_1(V)$ vanishes identically which is a stronger condition than $c_1(V) = 0 \pmod{2}$, the obstruction to guarantee that the bundle V admits spinors.

Finally, the dilatino transforms as

$$\delta\lambda = -\frac{1}{\sqrt{2}\phi}(\Gamma \cdot \partial\phi)\eta + \frac{\kappa}{8\sqrt{2}g^2\phi}\Gamma^{ijk}\eta H_{ijk} \quad (2.6)$$

and with the above ansatz of vanishing three form H and constant dilaton ϕ , $\delta\lambda = 0$ is satisfied. Furthermore, we have to consider the Bianchi identity for F and H which is essentially a topological condition and given by

$$dH = \text{tr} R \wedge R - \text{tr} F \wedge F \quad (2.7)$$

and simplifies in our case to $\text{tr} R \wedge R = \text{tr} F \wedge F$ or equivalently to

$$c_2(TK) = c_2(V) \quad (2.8)$$

where TK denotes the tangential bundle of K .

¹Let Z be a compact Kähler manifold of complex dimension d with Kähler form Ω and let V be a holomorphic rank n vector bundle on Z . One defines the normalized degree (or slope) of V with respect to Ω as the real number $\mu(V) = \frac{1}{\text{rank}(V)} \int_Z c_1(V) \wedge \Omega^{d-1}$. A holomorphic rank two vector bundle over Z is *stable* with respect to Ω (Ω -*stable*) if, for any line bundle F on Z there exists a nonzero map $F \rightarrow V$, one has $\mu(F) < \mu(V)$ and V is called *semistable* if $\mu(F) \leq \mu(V)$ holds. For arbitrary rank *stability* resp. *semistability* are similarly defined; one replaces F by a coherent analytic subsheaf \mathcal{F} of the sheaf of holomorphic sections of V such that $0 < \text{rank} \mathcal{F} < \text{rank} V$. However, we are interested in vector bundles over Calabi-Yau manifolds for which $\mu(V) = 0$. We need $H^0(Z, V) = H^3(Z, V) = 0$, which means that V has no sections. Otherwise, a nonzero element of $H^0(Z, V)$ defines a mapping from $\mathcal{O} \rightarrow V$ from the trivial line bundle into V but since the trivial line bundle has rank one, this would violate the inequality.

In summary, we have found that in order to compactify the heterotic string, one has to specify a Calabi-Yau manifold Z and a stable, holomorphic vector bundle $V \rightarrow Z$ satisfying the conditions

$$c_1(V) = 0, \quad c_2(V) = c_2(TZ). \quad (2.9)$$

Let V be a $SU(n)$ vector bundle over Z . The unbroken space-time gauge group will be the maximal subgroup of $E_8 \times E_8$ that commutes with the vacuum gauge field. For example, a $SU(n)$ vector bundle embedded in one E_8 , with $n = 3, 4$ or 5 , leads to an unbroken E_6 , $SO(10)$ resp. $SU(5)$ gauge group (times a "hidden" E_8 which couples only gravitationally).

The spectrum of charged matter is directly related to properties of Z and V . So, let us obtain the spectrum of massless fermions! We start in ten dimensions with the Dirac equation

$$iD_{10}\Psi = 0 = i(D_4 + D_Z)\Psi \quad (2.10)$$

further making the ansatz

$$\Psi = \psi(x)\phi(y) \quad (2.11)$$

where x and y are coordinates on Z respectively M_4 . If ψ is an eigenspinor of eigenvalue m then it follows

$$iD_Z\psi = m\psi \quad (2.12)$$

and so we get

$$(iD_4 + m)\phi = 0. \quad (2.13)$$

So we learn that ψ looks like a fermion of mass m , to a four-dimensional observer. Thus, massless four-dimensional fermions are in one to one correspondence with zero modes of the Dirac operator on Z . The charged four-dimensional fermions are obtained from ten-dimensional ones, which transform under the adjoint of E_8 .

Now since massless fermions in four dimensions are related to the zero modes of the Dirac operator on Z , they can be related to the cohomology groups $H^k(Z, V)$. Now, the index of the Dirac operator can be written as

$$\chi(V) = index(D_V) = \int_Z td(Z)ch(V) = \frac{1}{2} \int_Z c_3(V). \quad (2.14)$$

On the other hand we have

$$\chi(V) = \sum_{i=0}^3 (-1)^i h^i(Z, V) \quad (2.15)$$

and if we recall that $h^0(Z, V) = h^3(Z, V) = 0$ for stable bundles, we can write

$$h^2(Z, V) - h^1(Z, V) = \frac{1}{2} \int_Z c_3(V). \quad (2.16)$$

and by Serre duality on Z : $h^2(Z, V)$ is dual to $h^1(Z, V^*)$. The index measures the net number of generations N_{gen} of chiral fermions, thus we get

$$N_{gen} = \frac{1}{2} c_3(V) \quad (2.17)$$

In addition one has to take into account a number of bundle moduli $h^1(Z, \text{End}(V))$, which are gauge singlets.

2.2 Non-perturbative correction

So far we have reviewed the conditions for perturbative heterotic string compactifications. Now, Duff, Minasian and Witten have argued, that the perturbative anomaly cancellation condition can be modified in the presence of nonperturbative fivebranes in compactifications to six dimensions [38]

$$n_1 + n_2 + n_5 = c_2(Z). \quad (2.18)$$

Further, Friedman, Morgan and Witten [26] have shown that a similar relation appears in four-dimensional heterotic string compactifications on elliptically fibered Calabi-Yau threefolds. More precisely, the consistency of F-theory compactification on Calabi-Yau fourfolds requires the presence of a number n_3 of threebranes in the vacuum [25], which should turn into fivebranes under duality with the heterotic string [26] which wrap over fibers of $\pi : Z \rightarrow B_2$. The question was: why does a precise number of fivebranes appear in the heterotic vacuum? They showed that this is explained by anomaly cancellation. They considered an $E_8 \times E_8$ vector bundle $V = V_1 \times V_2$ which is determined by its fundamental characteristic class $\lambda(V_1)$ and $\lambda(V_2)$ ² with

$$\lambda(V_i) = \sigma \pi^*(\eta_i) + \pi^*(\omega) \quad (2.19)$$

² $\lambda(V) = \frac{1}{8\pi^2} \text{tr} F \wedge F$, taking the trace in the adjoint representation leads to $\lambda(V) = c_2(V)/60$

where $\eta_i \in H^2(B_2, \mathbf{Z})$ and $\omega \in H^4(B_2, \mathbf{Z})$, in particular, η_i is arbitrary but ω is determined in terms of η_i . Then, it was shown that the push-forward of $\lambda(V_i) = c_2(TZ)$ leads to (TZ denotes the tangent bundle to Z)

$$\pi_*(c_2(TZ)) - \pi_*(\lambda(V_1)) - \pi_*(\lambda(V_2)) = h[p]. \quad (2.20)$$

with $h \in \mathbf{Z}$ and $[p]$ is the class of a point and therefore the pullback of $\lambda(V_i) = c_2(TZ)$ is in error by a cohomology class of B_2 . This leads to the general anomaly cancellation condition with fivebranes

$$\lambda(V_1) + \lambda(V_2) + [W] = c_2(TZ) \quad (2.21)$$

where $[W] = h[F]$ denotes the cohomology class of the fivebranes and $[F] = \pi^*([p])$ is the class dual to the fibre.

Furthermore, it was shown [26] that a choice of η_i , leads to a prediction for h , which agrees with the number of F-theory threebranes n_3 and so $h = n_5$.

Note: We will presently show (c.f. section 5) that $n_3 = n_5$ holds also for general $SU(n)$ vector bundles!

3 Calabi-Yau spaces and Vector bundles

In section 3.1, we state a few facts about a class of elliptically fibered Calabi-Yau manifolds, which can be represented by a smooth Weierstrass model. These manifolds are used in the heterotic/F-theory duality.

In section 3.2, we construct a certain class of $SU(n)$ vector bundles, with n odd, and compute their third Chern-class $c_3(V)$, which is related to the net amount of chiral matter.

In section 3.3, we derive the number of bundle moduli for the E_8 bundle, for the tangent bundle of a smooth elliptic Calabi-Yau threefold, and review the computation for $SU(n)$ bundles, which has been performed in [26].

3.1 Elliptic fibrations

Let Z be an elliptic fibered Calabi Yau $\pi : Z \rightarrow B$ and let σ be a section of π such that $\pi \circ \sigma = id_B$.

In the following we are interested in the cases that B being complex 1, 2 or 3 dimensional denoted by B_1, B_2 and B_3 respectively which leads then to a elliptic fibered Calabi-Yau twofold (the $K3$), a Calabi-Yau three- respectively fourfold.

Z can be described by a Weierstrass equation

$$y^2z = x^3 + g_2xz^2 + g_3z^3 \quad (3.1)$$

which embeds Z in a \mathbf{P}^2 bundle $W \rightarrow B$; g_2 and g_3 are polynomials on the base. The \mathbf{P}^2 bundle is the projectivization of a vector bundle $\mathbf{P}(\mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \mathcal{O}_B)$ with \mathcal{L} being a line bundle over B . Since the canonical bundle of Z has to be trivial we get from $K_Z = \pi^*(K_B + \mathcal{L})$ the condition $\mathcal{L} = -K_B$. Further we can think of x, y and z as homogeneous coordinates on the \mathbf{P}^2 fibers, i.e. they are sections of $\mathcal{O}(1) \otimes K_B^{-2}$, $\mathcal{O}(1) \otimes K_B^{-3}$ and $\mathcal{O}(1)$. Furthermore g_2 and g_3 are sections of $H^0(B_3, K_B^{-4})$ and $H^0(B_3, K_B^{-6})$ respectively [39]. Thus Z as defined through the Weierstrass equation is a section of $\mathcal{O}(1)^3 \otimes K_B^{-6}$. The section σ can be thought of as the point at infinity in $\mathbf{P}(K_B^{-2} \oplus K_B^{-3} \oplus \mathcal{O}_B)$ defined by $x = z = 0, y = 1$.

Also we have to note that such an elliptic fibration has a \mathbf{Z}_2 symmetry which acts as multiplication by -1 on the canonical bundle ³ of the elliptic fibers of π while acting trivially on the canonical bundle of the base. If Z is represented as above, the \mathbf{Z}_2

³The canonical bundle of the elliptic fibers is generated by the differential form dx/y .

symmetry is generated by the transformation

$$\tau : y \rightarrow -y. \quad (3.2)$$

The discriminant Δ of π is given by

$$\Delta(g_2, g_3) = g_2^3 - 27g_3^2 \quad (3.3)$$

which is a section of K_B^{-12} . The j -invariant of Z

$$j_Z = 1728 \frac{g_2^3}{\Delta} \quad (3.4)$$

is a quotient of two sections of K_B^{-12} and thus a meromorphic function on B . For any point p on B with $\pi^{-1}(p) = E_p$ nonsingular fiber $j_Z(p) = j(E_p)$ is the usual j -invariant.

If $\Delta = 0$ then the elliptic fiber is singular. The type of singular fibre is determined by the order of vanishing of g_2 , g_3 and Δ at codimension one in the base. Let us denote by F, G and D the classes of the divisors associated to the vanishing of g_2, g_3 and Δ respectively, since X being Calabi-Yau we have $F = -4K_B$, $G = -6K_B$ and $D = -12K_B$.

3.1.1 Characteristic classes

As noted above x, y and z can be thought of as homogeneous coordinates on the \mathbf{P}^2 bundle W over B respectively as sections of line bundles whose first Chern classes we denote by $c_1(\mathcal{O}(1) \otimes K_B^{-2}) = r + 2c_1$, $c_1(\mathcal{O}(1) \otimes K_B^{-3}) = r + 3c_1$ and $c_1(\mathcal{O}(1)) = r$. The cohomology ring of W is generated over the cohomology ring of B by the element r with the relation $r(r + 2c_1(\mathcal{L}))(r + 3c_1(\mathcal{L})) = 0$ which expresses the fact that x, y and z have no common zeros.

Since Z is defined by the vanishing of a section of $\mathcal{O}(1)^3 \otimes K_B^{-3}$, which is a line bundle over W with first Chern class $3r + 6c_1$, any cohomology class on Z that can be extended to one over W , can be integrated over Z by multiplying it by $3r + 6c_1$ and then integrating over W , i.e. multiplication by $3(r + 2c_1)$ can be understood as restricting W to Z .

The relation $r(r + 2c_1(\mathcal{L}))(r + 3c_1(\mathcal{L}))$ can then be simplified to $r(r + 3c_1) = 0$ in the cohomology ring of Z . Now the total Chern class of the tangent bundle of W is

$$c(W) = c(B)(1 + r)(1 + r + 2c_1(\mathcal{L}))(1 + r + 3c_1(\mathcal{L})) \quad (3.5)$$

where $c(B)$ denotes the total Chern class of the tangent bundle of B . The total Chern class of Z is then obtained by dividing $c(W)$ by $(1 + 3r + 6c_1(\mathcal{L}))$

$$c(Z) = c(B) \frac{(1+r)(1+r+2c_1(\mathcal{L}))(1+r+3c_1(\mathcal{L}))}{1+3r+6c_1(\mathcal{L})} \quad (3.6)$$

Expanding $c(Z)$ leads to the Chern classes of Z . To obey the Calabi-Yau condition $c_1(\mathcal{L}) = c_1(B) = c_1$. For the cases of Z being complex 2, 3 or 4 dimensional (denoted by X_2, X_3, X_4) resp. with bases B_1, B_2 and B_3 we find (using $r^2 = -3rc_1$)

$$B_1 : c_2(X_2) = 4rc_1 \quad (3.7)$$

$$B_2 : c_2(X_3) = c_2 + 11c_1^2 + 4rc_1 \quad (3.8)$$

$$c_3(X_3) = -20rc_1^2 \quad (3.9)$$

$$B_3 : c_2(X_4) = c_2 + 11c_1^2 + 4rc_1 \quad (3.10)$$

$$c_3(X_4) = c_3 - 20rc_1^2 - c_1c_2 - 60c_1^3 \quad (3.11)$$

$$c_4(X_4) = 4rc_1c_2 + 120rc_1^3. \quad (3.12)$$

The Euler characteristic of X_i is given by

$$\chi(X_i) = \int_{X_i} c_i. \quad (3.13)$$

z is a section of the line bundle $\mathcal{O}(1)$ which is defined on the total space of W . The divisor $z = 0$ which is dual to the cohomology class r intersects the generic fiber E_p in three points, i.e. $r \cdot E_p = 3$. Hence, first intergrating over the fibers and then over B leads to the Euler characteristics for X_2, X_3 and X_4

$$\chi(X_2) = 12 \int_{B_1} c_1 \quad (3.14)$$

$$\chi(X_3) = -60 \int_{B_2} c_1^2 \quad (3.15)$$

$$\chi(X_4) = 12 \int_{B_3} c_1(c_2 + 30c_1^2). \quad (3.16)$$

Comment: Another way to derive $\chi(X_3)$ is as follows: Recall, that the discriminant Δ describes a codimension one sublocus in the base B . In case that B is two-dimensional, D is a curve in B_2 . The class of the curve is given by $D = -12K_{B_2}$. Furthermore, we have the classes $G = -4K_{B_2}$ and $F = -6K_{B_2}$. Now, we are interested in $\chi(X_3)$ which is described by a smooth Weierstrass model, that tells us that we can only expect type I (and II) singular fibers over D which contribute to $\chi(X_3)$. Now, the

singularity type of the singular fiber tells us the deviation of the Euler characteristic from the generic elliptic fiber, which has $\chi(T^2) = 0$. So, we should expect that $\chi(X_3) = \chi(\text{sing.fiber})\chi(D)$. Since D is a curve we have $-D(D + K_{B_2}) = -132c_1^2$ (where $c_i = c_i(B_2)$). But D itself can be singular! This happens at those points where the divisors associated to the classes F and G collide, and this happens at $F \cdot G = 24c_1^2$ points. At these points D develops a cusp and the elliptic fibre will be of type II. Using the standard Plücker formula, which takes the double points into account (c.f. [40]), we get $\chi(\tilde{D}) = -132c_1^2 + 2(24c_1^2)$, and for $\chi(X_3)$ we get

$$\begin{aligned}\chi(X_3) &= 1(-84c_1^2 - 24c_1^2) + 2(24c_1^2) \\ &= -60c_1^2.\end{aligned}\tag{3.17}$$

Remarks on the base: At this point we review some facts about B_3 , which we will need for establishing heterotic/F-theory duality following [26]. Let $\sigma : B_3 \rightarrow B_2$ be a \mathbf{P}^1 bundle which is the projectivization $\mathbf{P}(Y)$ of a vector bundle $Y = \mathcal{O} \oplus \mathcal{T}$, with \mathcal{T} a line bundle over B_2 and $\mathcal{O}(1)$ a line bundle on the total space of $\mathbf{P}(Y) \rightarrow B_2$ which restricts on each \mathbf{P}^1 fiber to the typical line bundle over \mathbf{P}^1 . Further let a, b be homogeneous coordinates of the \mathbf{P}^1 bundle and think of a and b as sections, respectively, of $\mathcal{O}(1)$ and $\mathcal{O}(1) \otimes \mathcal{T}$ over B_2 . If we set $r = c_1(\mathcal{O}(1))$ and $t = c_1(\mathcal{T})$ and $c_1(\mathcal{O} \otimes \mathcal{T}) = r + t$ then the cohomology ring of B_3 is generated over the cohomology ring of B_2 by the element r with the relation $r(r+t)=0$, i.e. the divisors $a = 0$ resp. $b = 0$, which are dual to r resp. $r + t$, do not intersect. The characteristic classes of B_3 can be derived from the total Chern class of B_3 given by

$$c(B_3) = (1 + c_1(B_2) + c_2(B_2))(1 + r)(1 + r + t)\tag{3.18}$$

one finds using $(c_i = \sigma^*(c_i(B_2)))$

$$c_1(B_3) = c_1 + 2r + t\tag{3.19}$$

$$c_2(B_3) = c_2 + c_1t + 2c_1r.\tag{3.20}$$

Furthermore, note that B_2 has to be rational which can be seen as follows: Recall the arithmetic genus p_a of B_3 has to be equal to one in order to satisfy the $SU(4)$ holonomy condition, otherwise there are non-constant holomorphic differentials on B_3 which would pull back to X and thus destroy $SU(4)$ holonomy of X , i.e. $p_a = \int_{B_3} c_1(B_3)c_2(B_3)/24 = 1$ and using the expressions for $c_1(B_3)$ and $c_2(B_3)$ one finds $p_a = \frac{1}{12} \int_{B_2} c_1^2 + c_2 = 1$ saying that B_2 is rational.

Further, it is known that for elliptic fibered Calabi-Yau threefolds X_3 the base B_2 has at worst log-terminal⁴ singularities. In particular one can consider [21] B_2 's which, after resolving of singularities, are either surfaces whose canonical bundle is trivial, i.e. $K3$ surface, Enriques surface, a hyperelliptic surface, a torus, or they are F_n , dP_k , $\frac{T^2 \times \mathbf{P}^1}{G}$. Here F_n is the Hirzebruch surface, a \mathbf{P}^1 bundle over \mathbf{P}^1 resp. the del Pezzo surface, get by blow up of \mathbf{P}^2 in $k = 1, \dots, k$ points.

Finally, let us recall that the expressions for the Euler numbers of X_3 and X_4 are given by [25],[44]

$$\chi(X_3) = 2(h^{1,1} - h^{2,1}) \quad (3.21)$$

$$\chi(X_4) = 6(8 + h^{1,1} + h^{3,1} - h^{2,1}). \quad (3.22)$$

All computations done so far are valid for elliptic fibrations which have a section and can be described by a smooth Weierstrass model.

⁴A variety X is said to have at worst *log-terminal* singularities, if the following conditions are satisfied:

- (i) K_X is an element of $Div(X) \otimes \mathbf{Q}$ (i.e. X is \mathbf{Q} -Gorenstein)
- (ii) for a resolution of singularities $\rho : Y \rightarrow X$ such that the exceptional locus of ρ is a divisor whose irreducible components $D_1 \dots D_r$ are smooth divisors with only normal crossings, one has $K_Y = \rho^* K_X + \sum_{i=1}^r a_i D_i$ with $a_i > -1$ (resp. $a_i \geq 0$) for all i , where D_i runs over all irreducible components of D [41],[42],[43].

3.2 Vector bundles on elliptic fibrations

To compactify the heterotic string on an elliptic Calabi-Yau threefold Z , one has to specify a vector bundle V over Z , which breaks part of the $E_8 \times E_8$ gauge symmetry. Friedman, Morgan and Witten have extensively studied the construction of vector bundles over Z using the parabolic and spectral cover approach for V [26], [35], [45].

In the parabolic approach they considered a component of the moduli space of $SU(n)$ bundles (understood as rank n complex vector bundles) which are τ -invariant, i.e. the involution of the elliptic fibration $Z \rightarrow B$ lifts up to V . This condition could be implemented for $SU(n)$ bundles with n even.

In the spectral cover approach they considered $V = SU(n)$ bundles with n arbitrary [26]. These bundles possess an additional degree of freedom since one has the possibility to 'twist' with a line bundle \mathcal{N} on the spectral cover C , which leads to a multi-component structure of the moduli space of such bundles. Further, they did not require any τ -invariance of V in the construction, moreover, they found τ -invariance for a certain class of these bundles which have no additional twists. In particular, they found the same modulo conditions (see below) as in the parabolic approach, in order to get τ -invariance. Further, their τ -invariant bundles have vanishing third Chern-class.

In the following section we will adopt these approaches. In contrast, we will not work at the τ -invariant point, we will construct [46] $SU(n)$ vector bundles using the parabolic approach which have n odd. We compare our bundles with the spectral cover approach and we find that for a certain 'twist' of the line bundle \mathcal{N} , both approaches agree. In particular the bundles have a non-vanishing third Chern-class which leads to a non-vanishing net number of chiral fermions in heterotic string compactifications.

3.2.1 Bundles from parabolics

In the parabolic approach one starts with an unstable bundle on a single elliptic curve E which is given by⁵

$$V = W_k \oplus W_{n-k}^* \quad (3.23)$$

this has the property that it can be deformed by an element $\alpha \in H^1(E, W_k^* \otimes W_{n-k}^*)$ to a (semi) stable bundle V' over E which fits in the exact sequence

$$0 \rightarrow W_{n-k}^* \rightarrow V' \rightarrow W_k \rightarrow 0 \quad (3.24)$$

⁵here W_k can be defined inductively as unique non-split extension $0 \rightarrow \mathcal{O} \rightarrow W_{k+1} \rightarrow W_k \rightarrow 0$ (c.f.[26])

Now, to get a global version of this construction one is interested in an unstable bundle over Z , which reduces on every fibre of $\pi : Z \rightarrow B$ to the unstable bundle over E and can be deformed to a stable bundle over Z . Since the basic bulding blocks were on each fiber W_k with $W_1 = \mathcal{O}(p)$, one has to replace them by their global versions. So, global one replaces $W_1 = \mathcal{O}(p)$ by $W_1 = \mathcal{O}(\sigma)$ and W_k can be defined inductively by an exact sequence

$$0 \rightarrow \mathcal{L}^{n-1} \rightarrow W_k \rightarrow W_{k-1} \rightarrow 0 \quad (3.25)$$

with $\mathcal{L} = K_B^{-1}$. Further, one has globally the possibility to twist by additional data coming from the base B and so one can write [26] for $V = SU(n)$

$$V = W_k \otimes \mathcal{M} \oplus W_{n-k}^* \otimes \mathcal{M}'. \quad (3.26)$$

Note that the unstable bundle can be deformed to a stable one by an element in $H^0(B, R^1\pi_*(ad(V)))$ since the Leray spectral sequence degenerates to an exact sequence [26]. Further, note that for the computation of characteristic classes we can use the unstable bundle because the topology of the bundle is invariant under deformations.

Now, let us start with the unstable $G = SU(n)$ bundle V where $\mathcal{M}, \mathcal{M}'$ are line bundles over B which are constrained so that V has trivial determinant, i.e. $\mathcal{M}^k \otimes (\mathcal{M}')^{n-k} \otimes \mathcal{L}^{-\frac{1}{2}(n-1)(n-2k)} \cong \mathcal{O}$. Further, W_k and W_{n-k}^* are defined as

$$W_k = \bigoplus_{a=0}^{k-1} \mathcal{L}^a, \quad W_{n-k}^* = \bigoplus_{b=0}^{n-k-1} \mathcal{L}^b \quad (3.27)$$

where we have set $\mathcal{L}^0 = \mathcal{O}(\sigma)$ and $\mathcal{L}^{-0} = \mathcal{O}(\sigma)^{-1}$ with \mathcal{L} being a line bundle on B . The total Chern class of V can then be written as

$$c(V) = \prod_{a=0}^{k-1} (1 + c_1(\mathcal{L}^a) + c_1(\mathcal{M})) \prod_{b=0}^{n-k-1} (1 + c_1(\mathcal{L}^{-b}) + c_1(\mathcal{M}')). \quad (3.28)$$

In the following we will discuss the two cases: n is even and n is odd. First let us review the case that n is even which was considered in [26], then we will extend to the case n is odd and compare both cases with the spectral cover construction of V .

n even

In this case one can choose $k = \frac{n}{2}$ which restricts one to take $\mathcal{M}' = \mathcal{M}^{-1}$ in order to obey trivial determinant of V . The advantage of taking $k = \frac{n}{2}$ is that the condition

of τ invariance of V is easily implemented. Note that τ operates on V as $\tau^*V = V^*$, i.e. $k \rightarrow n - k$. Now expansion of the total Chern class of V immediatly leads to $c_1(V) = 0$ and $c_3(V) = 0$. Further setting $\sigma = c_1(\mathcal{O}(\sigma))$ and $\eta = -2c_1(\mathcal{M}) + c_1(\mathcal{L})$ respectively using the fact that $\sigma^2 = -\sigma c_1(\mathcal{L})$, one obtains for the second Chern class

$$c_2(V) = \eta\sigma - \frac{1}{24}c_1(\mathcal{L})^2(n^3 - n) - \frac{n}{8}\eta(\eta - nc_1(\mathcal{L})). \quad (3.29)$$

Moreover, from $c_1(\mathcal{M}) = -\frac{1}{2}(\eta - c_1(\mathcal{L}))$ one gets the congruence relation for η :

$$\eta \equiv c_1(\mathcal{L}) \pmod{2} \quad (3.30)$$

Thus $c_2(V)$ is uniquely determined in terms of η and the elliptic Calabi-Yau manifold Z . In particular one has

$$\eta = \pi_*(c_2(V)). \quad (3.31)$$

Now let us turn to the case that n is odd!

n odd

Let us first specify [46] our unstable bundle V . Actually we have n different choices to do this which depend on the choice of the integer k in the range $1 \leq k \leq n$. We will choose $k = \frac{n+1}{2}$ and the line bundles $\mathcal{M} = \mathcal{S}^{-\frac{n+1}{2}}$ and $\mathcal{M}' = \mathcal{S}^{\frac{n+1}{2}} \otimes \mathcal{L}^{-1}$ which will be presently shown to be appropriate in order to compare it with results obtained from spectral covers for V . Therefore we can write for our unstable bundle V

$$V = W_k \otimes \mathcal{S}^{-n+k} \oplus W_{n-k}^* \otimes \mathcal{S}^k \otimes \mathcal{L}^{-1} \quad (3.32)$$

which has trivial determinant. Using the above relation for the total Chern-class of V and setting again $\sigma = c_1(\mathcal{O}(\sigma))$ and $\sigma^2 = -\sigma c_1(\mathcal{L})$, and

$$\eta = nc_1(\mathcal{S}) \quad (3.33)$$

we will find for the characteristic classes of V

$$c_1(V) = 0 \quad (3.34)$$

$$c_2(V) = \eta\sigma - \frac{1}{24}c_1(\mathcal{L})^2(n^3 - n) - \frac{n}{8}\eta(\eta - nc_1(\mathcal{L})) + \frac{1}{8n}\eta(\eta - nc_1(\mathcal{L})) \quad (3.35)$$

$$c_3(V) = \frac{1}{n}\sigma\eta(\eta - nc_1(\mathcal{L})) \quad (3.36)$$

So, we are restricted to bundles V with $\eta = \pi_*(c_2(V))$ divisible by n .

Now, integration of c_3 over Z can be accomplished by first integrating over the fibers of $Z \rightarrow B$ and then integrating over the base. Further, using the fact that the section σ intersects the fiber F in one point $\sigma \cdot F = 1$ and $r = 3\sigma$ where r is the cohomology class dual to the vanishing of the section of the line bundle $\mathcal{O}(1)$ (defined on the total space of the Weierstrass model of Z), we get

$$\int_Z c_3(V) = \int_B \frac{1}{n} \eta(\eta - nc_1(\mathcal{L})). \quad (3.37)$$

This leads to the net number N_{gen} of generations in a heterotic string compactification

$$N_{gen} = n_{gen} - \bar{n}_{gen} = \int_B \frac{1}{2n} \eta(\eta - nc_1(\mathcal{L})) \quad (3.38)$$

and thus, if we fix the elliptic manifold Z , so that the section σ and $c_1(\mathcal{L})$ are fixed, N_{gen} is uniquely determined for a choice of η . Here we have to note that in order to determine separately the number of generations n_{gen} respectively antigerations \bar{n}_{gen} instead of just their difference, we have in addition to compute the dimension of $H^1(Z, V)$. This can be done by using the Leray spectral sequence.

In order to compare our results with those obtained from the spectral cover construction let us review some facts of the setup [26].

3.2.2 Bundles from spectral covers

We are interested in $SU(n)$ vector bundles V over the elliptically fibered Calabi-Yau threefold $\pi : Z \rightarrow B$ with a section σ of $K_B = \mathcal{L}^{-1}$ and let Z be represented by its Weierstrass model, which embeds Z in a \mathbf{P}^2 bundle over B , as above, we have $y^2 = x^3 - g_2x - g_3$. In the spectral cover approach the vector bundle over Z is described by a pair (C, \mathcal{N}) , more precisely V can be reconstructed from this pair, where C is the spectral cover and \mathcal{N} an arbitrary line bundle over C . The spectral cover C is given by the vanishing of a section s of $\mathcal{O}(\sigma)^n \otimes \mathcal{M}$ with \mathcal{M} being an arbitrary line bundle over B of $c_1(\mathcal{M}) = \eta'$. The locus $s = 0$ for the section is given for n even by

$$s = a_0 z^n + a_2 z^{n-2} x + a_3 z^{n-3} y + \dots + a_n x^{n/2} \quad (3.39)$$

(resp. the last term is $x^{(n-3)/2} y$ for n odd)⁶ which determines a hypersurface in Z . If we fix a point in B then the Weierstrass equation and the locus $s = 0$ have n solutions

⁶here $a_r \in \Gamma(B, \mathcal{M} \otimes \mathcal{L}^{-r})$, a_0 is a section of \mathcal{M} and x, y sections of \mathcal{L}^2 resp. \mathcal{L}^3 in the Weierstrass model (c.f.[26])

and so C is an n -fold ramified cover of B . Further, one recovers V from C by choosing the Poincare line bundle \mathcal{P}_B over the fiber product $Z \times_B Z$ and considers its restriction to the subspace $Y = C \times_B Z$ of $Z \times_B Z$. Taking the projection to the second factor of the fiber product $\pi_2 : Y \rightarrow Z$, V can be constructed via push forward of the Poincare bundle on Y to Z . Further, taking into account that \mathcal{P}_B can be twisted by the pull back (via π_1) of the line bundle \mathcal{N} , so one has [26]

$$V = \pi_{2*}(\mathcal{N} \otimes \mathcal{P}_B) \quad (3.40)$$

It is important to note that \mathcal{N} is not completely arbitrary, it is restricted through the condition of vanishing first Chern-class of V . Since the Poincare line bundle becomes trivial when restricted to σ , one can use Grothendieck-Riemann-Roch for the projection $\pi : C \times_B \sigma = C \rightarrow B$ and gets

$$\pi_*(e^{c_1(\mathcal{N})}Td(C)) = ch(V)Td(B) \quad (3.41)$$

and with the condition $c_1(V) = 0$ one has

$$c_1(\mathcal{N}) = -\frac{1}{2}(c_1(C) - \pi_*c_1(B)) + \gamma = \frac{1}{2}(n\sigma + \eta' + c_1(\mathcal{L})) + \gamma \quad (3.42)$$

here $\gamma \in H^{1,1}(C, \mathbf{Z})$ with $\pi_*\gamma = 0 \in H^{1,1}(B, \mathbf{Z})$. In particular if one denotes by K_B and K_C the canonical bundles of B and C then one has [26]

$$\mathcal{N} = K_C^{1/2} \otimes K_B^{-1/2} \otimes (\mathcal{O}(\sigma)^n \otimes \mathcal{M}^{-1} \otimes \mathcal{L}^n)^\lambda \quad (3.43)$$

from which one learns that $\gamma = \lambda(n\sigma - \eta' + nc_1(\mathcal{L}))$.

The second Chern class of V is computed in [26] and given by

$$c_2(V) = \eta'\sigma - \frac{1}{24}c_1(\mathcal{L})^2(n^3 - n) - \frac{n}{8}\eta'(\eta' - nc_1(\mathcal{L})) - \frac{1}{2}\pi_*(\gamma^2) \quad (3.44)$$

where the last term reflects the fact that one can twist with a line bundle \mathcal{N} on the spectral cover, one has

$$\pi_*(\gamma^2) = -\lambda^2 n \eta'(\eta' - nc_1(\mathcal{L})). \quad (3.45)$$

For later use we will $c_2(V)$ denote (following [26]) by

$$c_2(V) = \sigma\eta + \omega \quad (3.46)$$

where $\omega \in H^4(B)$, i.e. $\omega = c_2(V|_B)$ and

$$\omega = -\left[\frac{n^3 - n}{6}\frac{c_1^2}{4} + \frac{n}{8}\eta(\eta - nc_1) + \frac{1}{2}\pi_*(\gamma^2)\right] \quad (3.47)$$

Now let us compare this with the parabolic computation we have done! Note that, $\lambda \neq 0$ (more precisely $\gamma \neq 0$) measures the deviation from τ -invariance for bundles in the spectral cover construction. We will start with the two cases $\lambda = 0$, $\lambda = \frac{1}{2}$ mentioned in [26] and then consider our case $\lambda = \frac{1}{2n}$.

In case that n is even, it was shown [26] that to achieve τ invariance in the spectral cover approach for V , one must define \mathcal{N} in the above sense with $\gamma = 0$, i.e. $\lambda = 0$. In particular the existence of an isomorphism $\mathcal{N}^2 = K_C \otimes K_B^{-1}$ was shown. Further, it was shown that there are the same mod two conditions for η' and n in the spectral cover approach for $\gamma = 0$ and in particular that also $\eta' = \pi_*(c_2(V))$ and therefore one is lead to the identification $\eta' = \eta$.

In case that n is even and $\lambda = \frac{1}{2}$ the last two terms in combine in (3.47) the only general elements of $H^{1,1}(C, \mathbf{Z})$ are $\sigma|_C$ and $\pi^*\beta$ (for $\beta \in H^{1,1}(B, \mathbf{Z})$), which have because of $C = n\sigma + \pi^*\pi_*c_2V$ the relation $\pi_*(\sigma|_C) = \pi_*\sigma(n\sigma + \pi^*\eta) = \pi_*\sigma(-nc_1 + \pi^*\eta) = \eta - nc_1$; so $\gamma = \lambda(n\sigma - \pi^*(\eta - nc_1))$ (with λ possibly half-integral) and $\pi_*(\gamma^2) = -\lambda^2 n\eta(\eta - nc_1)$; so for $\lambda = 1/2$ the term would disappear.

Now in our case, if n is odd, we can identify the last term in (3.35) with $\frac{\pi_*(\gamma^2)}{2}$ if we choose $\lambda = \frac{1}{2n}$, i.e. the parabolic approach for n odd agrees with the spectral cover approach if we choose the twisting line bundle \mathcal{N} appropriate on the spectral cover. Furthermore, if we use

$$c_1(\mathcal{N}) = \frac{1}{2}(n\sigma + \eta' + c_1(\mathcal{L})) + \gamma = \frac{(n+1)}{2}\sigma + c_1(\mathcal{L}) + \frac{(n-1)}{2}\frac{\eta'}{n} \quad (3.48)$$

which is well defined for n odd and since we can choose $\mathcal{M} = \mathcal{S}^n$ we are left with $\eta' = nc_1(\mathcal{S})$ and so with $\lambda = \frac{1}{2n}$ we have the same conditions for η' and n as we had in the parabolic approach.

Discussion: We have constructed a class of $SU(n)$ vector bundles, with n odd, in the parabolic bundle construction, which have a $\eta \equiv 0 \pmod{n}$ condition, in contrast to the bundles, which have n even and a $\eta \equiv c_1(\mathcal{L}) \pmod{2}$ condition. For n even, the bundles in the parabolic construction are restricted to the τ -invariant bundles in the spectral cover construction, given at $\lambda = 0$. Our bundles have no τ -invariance and being restricted to bundles of $\lambda = \frac{1}{2n}$, in the spectral cover construction.

Remark: In [47] the computation of $c_3(V)$ was performed in the spectral cover approach, it is

$$c_3(V) = 2\lambda\eta(\eta - nc_1). \quad (3.49)$$

and so in nice agreement with our computation of $c_3(V)$ in the parabolic approach at our special point $\lambda = \frac{1}{2n}!$

3.3 Bundle moduli

After describing the construction of vector bundles over an elliptic fibered Calabi-Yau manifold and computing the Chern classes of V we have now to determine an additional invariant characterizing our bundle, the dimension of $H^1(Z, \text{End}(V))$. This dimension will later play an important role, since the number of bundle moduli contributes to the massless spectrum in the heterotic string compactification.

In the first section of this chapter, we will review the computation for $Z = K3$ originally performed for the general case in [48]. Then we turn to the threefold case and count the moduli $h^1(Z, \text{ad}(V))$ for $V = E_8$ by applying [49] an index computation, which was first used in this specific form by Friedman, Morgan and Witten [26] for $V = SU(n)$.

3.3.1 Bundles over surfaces

Let us determine the number of bundle moduli in the case that Z is a $K3$ surface. We start with the Hirzebruch-Riemann-Roch formula

$$\begin{aligned}\chi(Z, \text{End}(V)) &= \int_Z td(Z) ch(\text{End}(V)) = \int_Z td(Z) ch(V) ch(V^*) \\ &= \int_Z -2nc_2(V) + \frac{n^2}{12}(c_1(Z)^2 + c_2(Z))\end{aligned}\tag{3.50}$$

Now, the Riemann-Roch formula gives

$$1 - h^{0,1}(Z) + h^{0,2}(Z) = \frac{1}{12} \int_Z c_1(Z)^2 + c_2(Z).\tag{3.51}$$

Further, with

$$\chi(Z, \text{End}(V)) = \sum_{i=0}^2 (-1)^i h^i(Z, \text{End}(V))\tag{3.52}$$

we find for the dimension of the moduli space setting $h^0(Z, \text{End}(V)) = 0$ and $h^2(Z, \text{End}(V)) = 0$ (in order to get a smooth moduli space [48], [50]) we get

$$h^1(Z, \text{End}(V)) = 2nc_2(V) + n^2 h^{0,1}(Z) - (n^2 - 1)(1 + h^{0,2})\tag{3.53}$$

using the fact that $h^{0,1}(Z) = 0$ and $h^{0,2}(Z) = 1$ for $K3$ we get

$$h^1(Z, \text{End}(V)) = 2(nc_2(V) + 1 - n^2)\tag{3.54}$$

3.3.2 Bundles over elliptic threefolds

Let us recall the setup of the index-computation in [26] concerning the contribution of the bundle moduli in the case of elliptic Calabi-Yau threefolds.

As the usual quantity suitable for index-computation $\sum_{i=0}^3 (-1)^i h^i(Z, \text{End}(V))$ vanishes by Serre duality, one has to introduce a further twist to compute a character-valued index. Now because of the elliptic fibration structure one has on Z the involution τ coming from the "sign-flip" in the fibers and we furthermore assume that at least at some point, at the τ -invariant point, in moduli space the symmetry can be lifted to an action on the bundle. In particular, the action of τ lifts to an action on the adjoint bundle $\text{ad}(V)$, which are the traceless endomorphisms of $\text{End}(V)$. If one projects onto the τ -invariant part of the index problem one has

$$I = - \sum_{i=0}^3 (-1)^i \text{Tr}_{H^i(Z, \text{ad}(V))} \frac{1+\tau}{2} = - \sum_{i=0}^3 (-1)^i h^i(Z, \text{ad}(V))_e \quad (3.55)$$

where the subscript "e" (resp. "o") indicates the even (resp. odd) subspaces of $H^i(Z, \text{ad}(V))$ under τ and we used

$$\text{Tr}_{H^i(Z, \text{ad}(V))} \frac{1+\tau}{2} = h^i(Z, \text{ad}(V))_e. \quad (3.56)$$

The character-valued index simplifies by the vanishing of the ordinary index to

$$I = -\frac{1}{2} \sum_{i=0}^3 (-1)^i \text{Tr}_{H^i(Z, \text{ad}(V))} \tau. \quad (3.57)$$

If we have an unbroken gauge group H , which is the commutator of the group G of V , then $I = n_e - n_o$ has to be corrected by $h_e^0 - h_o^0$ denoting by $n_{e/o}$ the number $h^1(Z, \text{ad}(V))_{e/o}$ of massless even/odd chiral superfields and by $h_{e/o}^0$ the number of unbroken gauge group generators even/odd under τ [26]. So one has for the number of bundle moduli

$$m_{\text{bun}} = h^1(Z, \text{ad}(V)) = n_e + n_o = I + 2n_o \quad (3.58)$$

Using a fixed point theorem, as it was extensively done in [26], one can effectively compute the character-valued index $I = n_e - n_o$ of bundle moduli

$$I = rk - \sum_j \int_{U_j} c_2(V) \quad (3.59)$$

where the U_j denotes the two fixed point sets which we want to describe explicitly in the following. Therefore recall that Z is described by a Weierstrass equation and τ is

the transformation $y \rightarrow -y$ which leaves the other coordinates fixed. At a fixed point the coordinates x , y and z are left fixed up to overall rescaling. From the Weierstrass equation

$$y^2 z = x^3 + g_2 x z^2 + g_3 z^3 \quad (3.60)$$

we see that there are two components of the fixed point set.

The first component, denoted by U_1 , is given in homogeneous coordinates by $(x, y, z) = (0, 1, 0)$ which is the standard section σ of $Z \rightarrow B$ and thus isomorphic to a copy of B and the cohomology class of U_1 is the class of the section.

The second component, denoted by U_2 , is given by the triple cover of B defined by

$$0 = x^3 + g_2 x z^2 + g_3 z^3 \quad (3.61)$$

which embeds U_2 in a \mathbf{P}^1 bundle $W \rightarrow B$. As already noted above, we can think of x and z as sections of $\mathcal{O}(1) \otimes K_B^{-2}$ and $\mathcal{O}(1)$ with first Chern classes $r + 2c_1$ and r respectively. The cohomology ring of W is generated over the cohomology ring of B by the element r with relation $r(r + 2c_1) = 0$, i.e. x and z have no common zeros or in other words the divisors dual to $x = 0$ and $z = 0$ do not intersect.

Now U_2 is defined by the vanishing of a section of $\mathcal{O}(1)^3 \otimes K_B^{-6}$ which is a line bundle over W with first Chern class $3r + 6c_1$ and multiplication by $3(r + 2c_1)$ can be understood as restricting W to U_2 . The relation $r(r + 2c_1) = 0$ can be simplified in the cohomology ring of U_2 to $r = 0$.

With this in mind, we are able to compute the index I using the identity (c.f.[49])

$$\int_{U_i} c_2(V) = \int_B c_2(V)|_{U_1} + 3 \int_B c_2(V)|_{U_2} \quad (3.62)$$

where 3 appears since U_2 is a triple cover of B (and for E_8 bundles replace $c_2(V)$ by $\lambda(V)$). Next let us determine the number of bundle moduli for E_8 and $SU(n)$ vector bundle!

Note: In [49] we performed the derivation in a slightly different way, using the non-perturbative anomaly cancellation condition and restricting $c_2(Z|_{U_i}) + [W]|_{U_i}$, to the fixed point set.

Bundle moduli for $V = E_8, SU(n)$:

The fundamental characteristic class $\lambda(V)^7$ of an E_8 vector bundle is given by [26]

$$\lambda(V) = \eta\sigma - 15\eta^2 + 135\eta c_1 - 310c_1^2 \quad (3.63)$$

${}^7\lambda(V) = c_2(V)/60$

which restricts to the fixed point set as

$$\int_B \lambda(V)|_{U_1} = \int_B (-\eta c_1 - 15\eta^2 + 135\eta c_1 - 310c_1^2) \quad (3.64)$$

$$3 \int_B \lambda(V)|_{U_2} = \int_B (-45\eta^2 + 405\eta c_1 - 930c_1^2) \quad (3.65)$$

and we will find for the index of the E_8 bundle using $\lambda(V)$ [49]

$$I = 8 - 4(\lambda(V) - \eta\sigma) + \eta c_1. \quad (3.66)$$

Further the fundamental characteristic class of an $SU(n)$ bundle is given by

$$c_2(V) = \eta\sigma + \omega \quad (3.67)$$

restriction to the fixed point set leads to

$$\int_B c_2(V)|_{U_1} = \int_B (-\eta c_1 + \omega) \quad (3.68)$$

$$3 \int_B c_2(V)|_{U_2} = \int_B (3\omega) \quad (3.69)$$

and the index of the $SU(n)$ bundle is given by [26]

$$I = n - 1 - 4(c_2(V) - \eta\sigma) + \eta c_1. \quad (3.70)$$

Let us also consider the case that $V = TZ$ the tangential bundle to Z [51]! Therefore recall the second Chern-class of Z was given by

$$c_2(Z) = c_2 + 11c_1^2 + 12\sigma c_1 \quad (3.71)$$

so, we find for the restriction to the fixed point set

$$\int_B c_2(Z)|_{U_1} = \int_B (c_2 - c_1^2) \quad (3.72)$$

$$3 \int_B c_2(Z)|_{U_2} = \int_B (3c_2 + 33c_1^2) \quad (3.73)$$

thus the index for the $SU(3)$ bundle is given by

$$\begin{aligned} I &= 2 - 4c_2 - 32c_1^2 \\ &= -46 - 28c_1^2. \end{aligned} \quad (3.74)$$

4 F-Theory

4.1 The framework

As already mentioned in the introduction, F-theory can be considered [21] as a twelve-dimensional theory underlying the conjectured $SL(2, \mathbf{Z})$ duality symmetry of type IIB in ten-dimensions.

Therefore recall, type IIB in ten-dimensions contains the Neveu-Schwarz-Neveu-Schwarz (NS-NS) sector with the bosonic fields g_{MN}, B_{MN}^A, ϕ and the Ramond-Ramond (R-R) sector with the bosonic fields $B_{MN}^P, \tilde{\phi}, A_{MNPQ}^+$, where ϕ is the dilaton and $\tilde{\phi}$ the axion which combine into $\tau = \tilde{\phi} + i \exp(-\phi)$. In particular, the equations of motion of low energy type IIB supergravity are covariant, and invariant under $SL(2, \mathbf{R})$ transformations [52],[53],[54]. It has been conjectured that $SL(2, \mathbf{R})$ is broken to its maximal compact subgroup $SL(2, \mathbf{Z})$ [6]. The metric and the four-form potential are invariant under $SL(2, \mathbf{Z})$, further, τ transforms as $\tau \rightarrow \frac{a\tau+b}{c\tau+d}$ where $(a, b, c, d) \in \mathbf{Z}$ with $ad - cb = 1$ and the antisymmetric tensor fields get exchanged. Thus τ transforms in the same way as the complex modulus of the torus. Now, in pure perturbative type IIB compactifications τ is constant. Here the F-theory starts! Compactifications of F-theory to lower dimensions can be formulated as type IIB theory on a manifold B with varying τ over B [21]. So, compactification of F-theory on a manifold X which admits an elliptic fibration over B can be considered as type IIB on B with $\tau = \tau(z)$ and $z \in B$. Regarding F-theory in this way avoids the difficulties in formulating a consistent twelve-dimensional theory. Now X can be represented by its Weierstrass model

$$y^2 = x^3 + xf(z_i) + g(z_i) \quad (4.1)$$

with f, g functions on the base B of degree 8 resp. 12. The elliptic fibre degenerates when $\Delta = 4f^3 + 27g^2$ vanishes. The class of the discriminant (as given above) is $D = -12K_B$, localized at codimension one in the base B . For example, if $X = K3$ the elliptic fibre degenerates over 24 points in $B = \mathbf{P}^1$, moreover, writing $\Delta = c \prod_{i=1}^{24} (z - z_i)$ and noting that $\tau \sim \frac{1}{2\pi i} \log(z - z_i)$, one finds that the τ modulus becomes singular if $z \rightarrow z_i$. If we go around $z = z_i$, we have $\tau \rightarrow \tau + 1$ and this implies a discrete shift of $\tilde{\phi} \rightarrow \tilde{\phi} + 1$, which has no explanation in perturbative type IIB compactification, in particular, this shift signals the presence of a magnetically charged 7-brane in $z = z_i$ filling the uncompactified spacetime [21] (also c.f. [55]).

The most simple degenerations of the elliptic fibre are of type I_1 (in the Kodaira classification) and the corresponding Weierstrass model is smooth. Unbroken gauge symmetry can be obtained in F-theory, if the elliptic fibre admits a section of higher singularities than I_1 , say a section of E_6 singularities, would then correspond in the conjectured het/F-theory duality to an unbroken E_6 space-time gauge group. Furthermore, on the compact part of the world-volume of the 7-brane one can turn on a non-trivial gauge field background with a nonzero instanton number [56]. The presence of such a background would then further break the gauge symmetry.

4.2 The constraints

In order to obtain consistent F-theory compactifications to four-dimensions on Calabi-Yau fourfold X , the necessity of turning on a number n_3 of space-time filling three-branes for tadpole cancellation was established [25]. This is motivated by the fact that compactifications of the type IIA string on X are destabilized at one loop by $\int B \wedge I_8$, where B is the NS-NS two-form which couples to the string and I_8 a linear combination of the Pontryagin classes p_2 and p_3 [24]. So, compactifications of the type IIA string to two-dimensions leads to a tadpole term $\int B$ which is proportional to the Euler characteristic of X . Similar, in M-theory compactifications to three-dimensions on X arises a term $\int C \wedge I_8$ with C being the M-theory three-form, and integration over X then leads to the tadpole term which is proportional to $\chi(X)$, and couples to the 2-brane [57],[58]. Now, as M-theory compactified to three-dimensions on X is expected to be related to a F-theory compactification to four-dimensions [21], one is lead to expect a term $\int A$ with A now being the R-R four form potential, which couples to the three-brane in F-theory [25]. Taking into account the proportionality constant [59], one finds $\frac{\chi(X)}{24} = n_3$ in F-theory (or $= n_2$ in M-theory resp. strings in type IIA theory) [25].

Furthermore, it was shown that the tadpole in M-theory will be corrected by a classical term $C \wedge dC \wedge dC$, which appears, if C gets a background value on X and thus leads to a contribution $\int dC \wedge dC$ to the tadpole [58]. Also, it has been shown [60] that one has as quantization law for the four-form field strength G of C (the four-flux) the modified integrality condition $G = \frac{dC}{2\pi} = \frac{c_2}{2} + \alpha$ with $\alpha \in H^4(X, \mathbf{Z})$ where α has to satisfy the bound [61] $-120 - \frac{\chi(X)}{12} \leq \alpha^2 + \alpha c_2 \leq -120$, in order to keep the wanted amount of supersymmetry in a consistent compactification. Also, the presence of a non-trivial instanton background can contribute to the anomaly [62]. Including all,

tadpole cancellation and the four-flux condition and non-trivial instanton background, one finds [63]

$$\frac{\chi(X)}{24} = n_3 + \frac{1}{2} \int G \wedge G + \sum_j \int_{\Delta_j} c_2(E_j) \quad (4.2)$$

for consistent $N = 1$ F-theory compactifications on X to four-dimensions where $\int_{\Delta_j} c_2(E_j) = k_j$ are the instanton numbers of possible background gauge bundles E_j inside the 7-brane [62] and Δ_j denotes the discriminant component in B_3 . Before going into details let us shortly recall how to establish the heterotic/F-theory duality!

4.3 The dualities

The basic idea for establishing a duality between two theories in lower dimensions is to use the *adiabatic principle* [24]. Therefore one has first to establish a duality between two theories, then one varies 'slowly' the parameters of these two theories over a common base space, and one expects that the duality holds on the lower dimensional space [21]. Furthermore, at those points where the adiabaticity breaks down, one expects new physics in the lower dimensional theory to come in.

Let us be more precise! The heterotic string compactified on a $n - 1$ -dimensional elliptically fibered Calabi-Yau $Z \rightarrow B$ together with a vector bundle V on Z is conjectured to be dual to F-theory compactified on a n -dimensional Calabi-Yau $X \rightarrow B$, fibered over the same base B with elliptic $K3$ fibers. A duality between the two theories involves the comparison of the moduli spaces on both sides. In the following we will be interested in $n = 2, 3, 4$.

Let us see how this works in various dimensions!

5 Heterotic/F-Theory duality

5.1 8D Het/F-duality

5.1.1 Heterotic string on T^2

Let us compactify the heterotic string on T^2 . Let us represent T^2 by its Weierstrass equation, i.e. a cubic in \mathbf{P}^2

$$y^2 = x^3 + g_2x + g_3 \quad (5.1)$$

In addition one has to specify a vector bundle on T^2 . This can be done by turning on Wilson lines on T^2 . In particular, turning on 16 Wilson lines on T^2 , breaks $E_8 \times E_8$ to $U(1)^{16}$, otherwise, we are left with the unbroken 10D gauge group, which involves a description of $E_8 \times E_8$ bundles on T^2 . It has been shown [26] that E_8 bundles on Z can be described by embedding Z in a rational elliptic surface, which can be obtained by blowing up 9 points in \mathbf{P}^2 .

Now, on the level of parameter counting one gets 16 complex parameters coming from the Wilson lines, and additional 2 complex parameters from the complex structure modulus U and the Kähler modulus T of T^2 . These 18 parameters parametrize the moduli space [64],[65]

$$\mathcal{M}_{het} = SO(18, 2; \mathbf{Z}) \backslash SO(18, 2) / SO(18) \times SO(2) \quad (5.2)$$

further, one has to take into account the heterotic coupling constant, which is parametrized by a positive real number λ^2 [21], so we have 18 complex + 1 real parameters.

5.1.2 F-theory on $K3$

Now, in [21] it was argued (on the level of parameter counting) that the heterotic string on T^2 in the presence of Wilson lines is dual to F-theory compactified on $K3$ given by

$$y^2 = x^3 + g_2(z)x + g_3(z) \quad (5.3)$$

where the equation describes a hypersurface in a \mathbf{P}^2 bundle over \mathbf{P}^1 . In particular one has $(9+13-3-1) = 18$ parameters, where 9+13 coming from specifying g_2 and g_3 , then mod out by $SL(2, \mathbf{C})$ action on \mathbf{P}^1 means just subtracting 3 and an additional 1 for

overall rescalings. Furthermore, the remaining real parameter (the heterotic coupling) can be identified with the size of the \mathbf{P}^1 .

$$\mathcal{M}_F = SO(18, 2; \mathbf{Z}) \backslash SO(18, 2) / SO(18) \times SO(2) \quad (5.4)$$

Further, it was argued [22] that, in case of switching off Wilson lines, the F-theory dual is given by the two-parameter family of $K3$'s

$$y^2 = x^3 + \alpha z^4 x + (z^5 + \beta z^6 + z^7) \quad (5.5)$$

with E_8 singularities at $z = 0, \infty$. Therefore, it must exist a map that relates the T and U modulus of the heterotic string to the two complex structure moduli α and β . This map was made explicit in [66] using the relationship between the $K3$ discriminant and the discriminant of the Calabi-Yau threefold $X_{1,1,2,8,12}(24)$ in the limit of large base \mathbf{P}^1 .

5.2 6D Het/F-duality

This section contains a review of some aspects⁸ of 6D het/F-duality, however, it is devoted to providing us with the necessary 'six-dimensional' tools to be used in section 6.3 and 6.4 where we will construct four-dimensional dual heterotic/F-theory pairs with $N = 1$ supersymmetry, by a \mathbf{Z}_2 modding of $N = 2$ models which are obtained from 6d ones by T^2 compactification.

5.2.1 Heterotic string on $K3$

We will compactify the heterotic string from ten to six-dimensions on $(K3, V)$ with $V = SU(n)$ vector bundle, which leads to $N = 1$ supergravity in six-dimensions. We start with the heterotic string on T^2 in eight-dimensions and vary over an additional \mathbf{P}^1 so that the family of T^2 's fits into an elliptically fibered $K3$, i.e. the elliptic fibers degenerate over 24 points in the base \mathbf{P}^1 . The singularity type of the elliptic fibre should not be worse than I_1 in the Kodaira classification, in order to obtain a smooth Weierstrass model. We can describe the elliptic $K3$ by its Weierstrass equation

$$y^2 = x^3 + g_2(z)x + g_3(z) \quad (5.6)$$

where g_2 and g_3 are the polynomials of degree 8 resp. 12 on the base.

The six-dimensional massless spectrum contains the supergravity multiplet G_6 with a graviton $g_{\mu\nu}$, a Weyl gravitino ψ_μ^+ and a self-dual anti-symmetric tensor $B_{\mu\nu}^+$; the hypermultiplet H includes a Weyl fermion χ^- and four real massless scalar fields φ ; the tensor multiplet T with an self-dual antisymmetric tensor $B_{\mu\nu}^-$, a real massless scalar field ϕ and a Weyl spinor ψ^- ; finally, the Yang-Mills multiplet V contains a vector A_μ and a gaugino λ^+ .

- $G_6 = (g_{\mu\nu}, \psi_\mu^+, B_{\mu\nu}^+) : (1, 1) + 2(\frac{1}{2}, 1) + (0, 1)$
- $H = (\chi^-, 4\varphi) : 2(\frac{1}{2}, 0) + 4(0, 0)$
- $T = (B_{\mu\nu}^-, \phi, \psi^-) : (1, 0) + (0, 0) + 2(\frac{1}{2}, 0)$
- $V = (A_\mu, \lambda^+) : (\frac{1}{2}, \frac{1}{2}) + 2(0, \frac{1}{2})$

where the massless representations of $N = 1$ supersymmetry are labeled by their $Spin(4) \sim SU(2) \times SU(2)$ representations.

⁸We will not focus on nonperturbative effects in 6d, such as, strong coupling singularities and tensionless strings.

We get moduli fields from hyper and tensor multiplets. Therefore one expects the moduli space to be in the form

$$\mathcal{M} = \mathcal{M}_H \times \mathcal{M}_T \quad (5.7)$$

where \mathcal{M}_H is a quaternionic Kähler manifold and \mathcal{M}_T is a Riemannian manifold, their dimensions are given below.

Since the supergravity is chiral, there are constraints on the allowed spectrum, due to gauge and gravitational anomaly cancellation conditions. The anomaly can be characterized by the exact anomaly eight-form $I_8 = dI_7$, given by [67]

$$I_8 = \alpha \text{tr} R^4 + \beta (\text{tr} R^2)^2 + \gamma \text{tr} R^2 \text{tr} F^2 + \delta (\text{tr} F^2)^2 \quad (5.8)$$

where F is the Yang-Mills two-form, R is the curvature two-form and $\alpha, \beta, \gamma, \delta$ are real coefficients depending on the spectrum of the theory. The gravitational anomaly can only be cancelled if one requires $\alpha = 0$, which leads to [68] the spectrum constraint

$$n_H - n_V + 29n_T = 273 \quad (5.9)$$

where n_H and n_V denote the total number hyper multiplets and vector multiplets respectively. In order to employ a Green-Schwarz mechanism to cancel the remaining anomaly, the remaining anomaly eight-form has to factorize, $I_8 \sim I_4 \wedge \tilde{I}_4$ with

$$I_4 = \text{tr} R^2 - \sum_i v_i (\text{tr} F^2)_i, \quad \tilde{I}_4 = \text{tr} R^2 - \sum_i \tilde{v}_i (\text{tr} F^2)_i \quad (5.10)$$

where v_i, \tilde{v}_i are constants which depend on the gauge group [68],[69],[70].

Further, if we denote the heterotic string coupling by $\lambda^2 = e^{2\phi}$, then the fact that the anomaly eight-form factorizes, implies [69] that the gauge kinetic term contains a term of the form

$$\mathcal{L} \propto (v e^{-\phi} + \tilde{v} e^{\phi}) \text{tr} F^2 \quad (5.11)$$

which at finite values of the heterotic coupling $e^{-2\phi} = \frac{-\tilde{v}}{v}$ becomes zero and therefore leads to a phase transition.

To cancel the remaining anomaly, the Green-Schwarz mechanism is employed by defining a modified field strength H for the antisymmetric tensor $H = dB + \omega^L - \sum_i v_i \omega_i^{YM}$ with $\int_{K3} dH = 0$ and $dH = I_4$. ω^L denotes the Lorentz-Chern-Simons form and ω_i^{YM} is the Yang-Mills Chern-Simons form. So, one gets the condition

$$\sum_i n_i = \sum_i \int_{K3} (\text{tr} F^2)_i = \int_{K3} R^2 = 24 \quad (5.12)$$

Turning on the instanton numbers n_i ($i = 1, 2$) in both E_8 's, with

$$n_1 + n_2 = 24, \quad (5.13)$$

breaks the gauge group to some subgroup $G_1 \times G_2$ of $E_8 \times E_8$.

In addition, one has the possibility of turning on a number n_5 of five-branes [71],[38] which are located at points on $K3$ where each point contributes a hypermultiplet (i.e. the position of each five-brane is parametrized by a hypermultiplet). The occurrence of nonperturbative five-branes in the vacuum then leads to the generalized anomaly cancellation condition

$$n_1 + n_2 + n_5 = 24. \quad (5.14)$$

Further, the occurrence of such five-branes changes the number of tensor multiplets in the massless spectrum, since on the world sheet theory of the five-brane a massless tensor field lives, one has

$$n_T = 1 + n_5. \quad (5.15)$$

So, the coulomb branch of the six-dimensional theory is parametrized by the scalar field vev's of the tensor multiplets. If $n_5 = 0$, then we have a purely perturbative heterotic compactification with one tensor multiplet, where the corresponding scalar field is the heterotic dilaton.

In order to obtain the complete massless spectrum, we have also to take into account the number of bundle moduli from compactification on $K3$, since they lead to additional gauge neutral hypermultiplets. In particular the number of bundle moduli for $V = SU(n)$ is given by (c.f. section 3)

$$h^1(K3, \text{End}(V)) = 2 \int_{K3} c_2(V) n + 1 - n^2 \quad (5.16)$$

So, the quaternionic dimension of the instanton moduli space of k instantons is given by

$$\begin{aligned} \dim_Q(\mathcal{M}_{inst}) &= nc_2(V) - (n^2 - 1) \\ &= c_2(H)k - \dim H \end{aligned} \quad (5.17)$$

where $c_2(H)$ is the dual Coxeter number of the commutant H in $G \times H \in E_8$. Thus the Higgs branch is parametrized by n_H hypermultiplets

$$n_H = 20 + n_5 + \dim_Q(\mathcal{M}_{inst}) \quad (5.18)$$

Now, if E_8 is broken to the subgroup G by giving gauge fields on $K3$ an expectation value in H , i.e. $G \times H \in E_8$ is a maximal subgroup, then the part of the spectrum arising from the Yang-Mills multiplet can be determined as follows:

The number of vector multiplets is given by the dimension of the adjoint representation of the gauge group, since the vector multiplets belong to the adjoint representation, so we have

$$n_V = \dim(\text{adj}(G)). \quad (5.19)$$

Further, the hyper multiplets belong to some representation M_i of G . Note that CPT invariance requires that M_i is real. Denoting by R_i the representations in H in the decomposition $\text{adj}(E_8) = \sum_i (M_i, R_i)$.

To determine the number of charged hypermultiplets, we consider an H -bundle V with fibre in an irreducible representation R_i of the structure group. Form the Dolbeault index on $K3$ on gets [72], [68]

$$\begin{aligned} \chi(T, V) &= \sum_{i=0}^2 (-1)^i h^i(T, V) = \int_{K3} td(T) \wedge ch(V) \\ &= 2 \dim(R_i) - \int_{K3} c_2(V) \text{index}(R_i). \end{aligned} \quad (5.20)$$

where T denotes the tangent bundle of $K3$. Since our bundle is stable, we have $h^0(T, V) = 0$ and by Serre duality $h^2(T, V) = 0$ or, to say it differently, if V admits nonzero global sections then the structure group would be a subgroup of H , but we are interested in strict H bundles. So, we get ⁹

$$-h^1(T, V) = 2 \dim(R_i) - \int_{K3} c_2(V) \text{index}(R_i) \quad (5.21)$$

which leads to a condition (c.f.[73]) for any irreducible representation R_i

$$c_2(V) \geq \frac{2 \dim(R_i)}{\text{index}(R_i)}. \quad (5.22)$$

Thus, for the number of hypermultiplets in the representation M_i , we get (count quaternionic)

$$N_{M_i} = \frac{1}{2} \int_{K3} c_2(V) \text{index}(R_i) - \dim(R_i) \quad (5.23)$$

⁹Note that, in the spectral cover construction of V one has in the case of $K3$ the class of the spectral curve given by $C = n\sigma - \pi^*\eta = n\sigma - \pi^*\pi_*c_2(V) = n\sigma - c_2(V)E$. The intersection number of C with σ of $Z \rightarrow P^1$ is given by $C \cdot \sigma = n\sigma^2 - c_2(V)E\sigma$ and recall that $E \cdot \sigma = 1$ and $\sigma^2 = -\sigma c_1(\mathcal{L}) = -2$, we find that $C \cdot \sigma = c_2(V) - 2n$. We thus learn that $h^1(V)$ gets a contribution whenever C intersects σ .

Examples: We start with $E_8 \times E_8$ heterotic string on $K3$ with $SU(2)$ bundles with instanton numbers $(n_1, n_2) = (12 - n, 12 + n)$ embedded into the E_8 's respectively, where $n_1 + n_2 = 24$. If $0 \leq n \leq 8$ then the resulting gauge group is $E_7 \times E_7$ and with the above index one computes the number $N_{\mathbf{56}}$ of massless hypermultiplets

$$\frac{1}{2}(8 - n)(\mathbf{56}, \mathbf{1}) + \frac{1}{2}(8 + n)(\mathbf{1}, \mathbf{56}). \quad (5.24)$$

In addition one has a number of gauge neutral hypermultiplets coming from the bundle moduli

$$\begin{aligned} \dim_Q(\mathcal{M}_{inst}) &= \dim_Q(\mathcal{M}_{inst}^{n_1}) + \dim_Q(\mathcal{M}_{inst}^{n_2}) \\ &= (21 + 2n) + (21 - 2n) \\ &= 42 \end{aligned} \quad (5.25)$$

and $h^{1,1}(K3) = 20$ universal hypermultiplets of $K3$, thus leading to $62(\mathbf{1}, \mathbf{1})$ gauge singlets. It was shown [22], [23], [74] that complete Higgsing of $E_7 \times E_7$ is possible for $n = 0, 1, 2$, in particular, the cases $n = 0$ and $n = 2$ are equivalent. The case $n = 12$ is the standard embedding, the tangential bundle embedded into one E_8 , with $c_2(V) = c_2(K3) = 24$, one gets $E_8 \times E_7$ with the massless hypermultiplets

$$10(\mathbf{1}, \mathbf{56}) + 65(\mathbf{1}, \mathbf{1}) \quad (5.26)$$

In this case, we can completely Higgs the E_7 getting $10 \cdot 56 - 133 = 427$ hypermultiplets, leading to 492 gauge neutral hypermultiplets and an unbroken E_8 gauge group, thus we get

$$\begin{aligned} n_T &= 1 \\ n_H &= 492 \\ n_V &= 248 \end{aligned} \quad (5.27)$$

Let us turn to the F-theory side!

5.2.2 F-theory on Calabi-Yau threefold

Let us consider F-theory on Calabi-Yau threefold. We start with F-theory in eight-dimensions on an elliptic $K3 \rightarrow \mathbf{P}_f^1$ [21]. Variation of the eight-dimensional data over an additional \mathbf{P}_b^1 leads to a family of elliptic $K3$ surfaces, which should then fit into a Calabi-Yau threefold X_3 .

More precisely, X_3 should admit an elliptic fibration as well as a $K3$ fibration, in particular, the generic $K3$ fibre should agree in the large volume limit of \mathbf{P}_b^1 with the heterotic $K3$ in six-dimensions. Thus X_3 must be an elliptic fibration over a two-dimensional base B_2 which itself has a \mathbf{P}_f^1 fibration over \mathbf{P}_b^1 , i.e. over the rational ruled Hirzebruch surface F_n [22], [23].

In the following, we will denote the elliptic fibered Calabi-Yau threefold over F_n by X_n^3 and denoting by z_1 and z_2 the complex coordinates of \mathbf{P}_b^1 respectively \mathbf{P}_f^1 .

The Weierstrass model for X_n^3 can be written as [22], [23]:

$$X_n^3: \quad y^2 = x^3 + \sum_{k=-4}^4 f_{8-nk}(z_1) z_2^{4-k} x + \sum_{k=-6}^6 g_{12-nk}(z_1) z_2^{6-k}. \quad (5.28)$$

where $f_{8-nk}(z_1)$, $g_{12-nk}(z_1)$ are polynomials of degree $8 - nk$, $12 - nk$ respectively, and the polynomials with negative degrees are identically set to zero. So we see that the Calabi-Yau threefolds X_n^3 are $K3$ fibrations over \mathbf{P}_b^1 with coordinate z_1 ; the $K3$ fibres themselves are elliptic fibrations over the \mathbf{P}_f^1 with coordinate z_2 .

The Hodge numbers $h^{(2,1)}(X_n^3)$, of X_n^3 , are then given by the number of parameters of the curve (5.28) minus the number of possible reparametrizations. The Hodge numbers $h^{(1,1)}(X_n^3)$ are determined by the Picard number ρ of the $K3$ -fibre of X_n^3 as

$$h^{(1,1)}(X_n^3) = 1 + \rho. \quad (5.29)$$

Furthermore, the heterotic string coupling was identified in the eight-dimensional duality with the size k_f of the \mathbf{P}_f^1 . Now, in six-dimensions one has [22],[23]

$$e^{-2\phi} = \frac{k_b}{k_f} \quad (5.30)$$

where k_b denotes the Kähler class of the base \mathbf{P}_b^1 . In particular, it was shown [22] that there is a bound for the heterotic coupling constant due to the Kähler cone of the Hirzebruch surface F_n , given by

$$\frac{k_b}{k_f} \geq \frac{n}{2} \quad (5.31)$$

Since on the heterotic side there occur phase transitions at $\frac{-\tilde{v}}{v}$, one was led to identify [22]

$$\frac{-\tilde{v}}{v} = \frac{n}{2} \quad (5.32)$$

This then lead to the duality conjecture between the heterotic string on $K3$ with an instanton distribution $(12 + n, 12 - n)$ between both E_8 's and F-theory compactified on Calabi-Yau threefold with F_n as base [21]. To test these dualities one has to assume that, if the heterotic string has an unbroken gauge group G then on the F-theory side the elliptic fibration must have a singularity of type G^{10} .

Let us now recall [21],[22],[23] how the Hodge numbers of X_n^3 determine the spectrum of the F -theory compactifications.

The number of tensor multiplets n_T is given by the number of Kähler deformations of the two dimensional type IIB base B_2 except for the overall volume of B_2 , one has

$$n_T = h^{11}(B_2) - 1. \quad (5.33)$$

Since tensor fields become abelian $N = 2$ vector fields upon further compactification to four-dimensions on T^2 and the four-dimensional F-theory becomes equivalent to type IIA on X_3 , one is lead to the number of four-dimensional abelian vector fields in the Coulomb phase $n_T + r(V) + 2 = h^{11}(X_3)$, where $r(V)$ denotes the rank of the six-dimensional gauge group and the 2 arises from the T^2 compactification. This then leads to:

$$r(V) = h^{11}(X_3) - h^{11}(B_2) - 1. \quad (5.34)$$

The number of hyper multiplets, which are neutral under the abelian gauge group, is given in four as well as in six-dimensions by:

$$n_H = h^{21}(X_3) + 1 \quad (5.35)$$

where the 1 is coming from the freedom in varying the overall volume of B_2 . These numbers are constrained by the anomaly cancellation condition

$$n_H - n_V + 29n_T = 273 \quad (5.36)$$

¹⁰Note that G is always simply laced (ADE groups). But if there is a monodromy action on the singularity, which is an automorphism of the root lattice, then the singularity does not correspond to these simply laced groups but to a quotient of them and these groups are non-simply laced. These singularities are called non-split (in contrast, they are called split, if G has only inner automorphisms)[75],[76].

which provides a check in the spectrum of matter in F-theory.

Example 1: On the heterotic side we had $21 + 2n$ instantons in one E_8 (resp. $21 - 2n$ in the other E_8). To obtain these numbers from F-theory we concentrate at gauge groups, appearing at $z_1 = 0$. From the Kodaira classification we find, that the polynomials f and g have to vanish on the base to orders 3 and 5. One finds [22] the equation $y^2 = x^3 + x(z^4 f_8 + z^3 f_{8+n}) + (\dots + z^6 g_1 2 + z^5 g_{12+n})$ and counting complex deformations preserving the singular locus leads to $(13 + n) + (9 + n) - 1 = 21 + 2n$ parameters in agreement with the number of neutral hypermultiplets (Note that the -1 comes from the rescaling of the z_1 coordinate). Further, we can read of charged matter from the discriminant near the E_7 locus $z_1 = 0$, which is given by $\Delta = z_1^9(4f_{8+n}^3(z_2) + o(z_1))$. Thus we see that f vanishes on $z_1 = 0$ at $8 + n$ points, corresponding to $8 + n$ 7-branes intersecting the one corresponding to E_7 ¹¹. So one expects that each charged $\frac{1}{2}$ -hypermultiplet in the **56** is localized at the points of collision of the divisors¹².

Example 2: The standard embedding on the heterotic side ($n = 12$) leads to $E_8 \times E_7$ and complete higgsing of E_7 leads to an unbroken E_8 . To obtain E_8 gauge symmetry from F-theory on $X_3 \rightarrow F_{12}$, we need a section $\theta : F_{12} \rightarrow X_3$ of E_8 singularities along a codimension one locus in F_{12} , i.e. we need type II^* singular fibers along a curve in the Hirzebruch surface F_{12} . In F_n we have two 'natural' curves given by the zero section S_0 and the section at infinity $S_\infty = S_0 + nf$ of the \mathbf{P}^1 bundle. Let us localize the II^* fibers along S_0 with $S_0^2 = -n$ (also one has $S_0 \cdot f = 1$). Now, we can decompose the discriminant Δ into two components: $\Delta = \Delta_1 + \Delta_2$, where Δ_1 denotes the component with I_1 fibers and Δ_2 has II^* fibres. Each component is characterized by the order of vanishing of some polynomials as Δ itself. Recall that Δ has the class $\Delta = -12K_B$, resp. $F = -4K_B$ and $G = -6K_B$. Similarly we can denote the class of Δ_2 by $\Delta_2 = 10S_0$, resp. $F_2 = 4S_0$ and $G_2 = 5S_0$. Thus, Δ_1 has the class $\Delta_1 = \Delta - \Delta_2$, resp. $F_1 = F - F_2$ and $G_1 = G - G_2$. With the canonical bundle of the Hirzebruch surface F_n , $K_{F_n} = -2S_0 - (2 + n)f$ we get (set $n = 12$): $\Delta_1 = 14S_0 + 168f$, resp. $F_1 = 4S_0 + 56f$ and $G_1 = 7S_0 + 84f$, so describing the locus of I_1 singularities. Further, we have $\Delta_1 \cdot \Delta_2 = 0$ which means that the two components do not intersect, otherwise

¹¹Note that the types of singularities over these extra branes do not necessarily correspond to an extra gauge group enhancement.

¹²Katz and Vafa [32] derived, purely in F-theory, the charged matter content without making any use of duality with the heterotic string.

we must blow-up within the base in order to produce a Calabi-Yau threefold [77]. Since we have no additional blow-up's in the base, we get $h^{1,1}(B_2) = 2$ (one from the base \mathbf{P}^1 and one from the fiber \mathbf{P}^1 in F_n), thus we get $n_T = 1$. The number of Kähler deformations of X_3 are given by $h^{1,2}(X_3) = 2 + 8 + 1 = 11$ (2 from the base, 8 from the resolution of the II^* singular fibers (c.f.[78])). The number of complex structure deformations is now equals $h^{2,1}(X_3) = h^{1,1}(X_3) - \frac{1}{2}\chi(X_3)$. Since only the singular fibers contribute to $\chi(X_3)$, we get $\chi(X_3) = \chi(\Delta_1)\chi(\text{I}_1) + \chi(S_0)\chi(\text{II}^*)$. Now Δ_1 is a curve in the base, which has cusp singularities at $F_1 \cdot G_1 = 392$ points, so applying the standard Plücker formula, we find $-\Delta_1(\Delta_1 + K_{F_{12}}) + 2(392) = -1372$. To determine $\chi(\Delta_1)$ we have to be careful with the cusp contribution, since over any cusp the elliptic fiber is of type II . Thus each cusp contributes with $\chi(\text{II})\chi(F_1 \cdot G_1)$ to $\chi(X_3)$. Therefore one gets, the corrected formula

$$\begin{aligned}\chi(X_3) &= \chi(\text{I}_1)(\chi(\Delta_1) - \chi(F_1 \cdot G_1)) + \chi(\text{II})\chi(F_1 \cdot G_1) + \chi(S_0)\chi(\text{II}^*) \\ &= -960\end{aligned}\tag{5.37}$$

using the fact that $\chi(S_0)\chi(\text{II}^*) = 2 \cdot 10$, where the 2 is the Euler characteristic of $S_0 = \mathbf{P}^1$ of F_n and the $\chi(\text{II}^*) = 2 \cdot 9 - 8 = 10$ (from the resolution tree of II^* singularity, one has 9 \mathbf{P}^1 's intersecting in 8 points (c.f.[78])). Thus we get the number of complex deformations $h^{2,1} = 491$ and so $n_H = 492$ hypermultiplets. So we end up with the F-theory spectrum

$$\begin{aligned}n_T &= 1 \\ n_H &= 492 \\ n_V &= 248.\end{aligned}\tag{5.38}$$

5.3 4D Het/F-duality

The four-dimensional heterotic/F-theory duality picture can be established by considering Z as an elliptic fibration $\pi : Z \rightarrow B_2$ with a section σ , where B_2 is a twofold base; Z can be represented as a smooth Weierstrass model. X was considered as being elliptically fibered over a threefold base B_3 , which is rationally ruled, i.e. there exists a fibration $B_3 \rightarrow B_2$ with \mathbf{P}^1 fibers because one has to assume the fourfold to be a $K3$ fibration over the twofold base B_2 in order to extend adiabatically the 8D duality between the heterotic string on T^2 and F-theory on $K3$ over the base B_2 .

During investigating F-theory compactifications with $N = 1$ supersymmetry in four dimensions on a Calabi-Yau fourfold X there was established the necessity of turning on a number $n_3 = \chi(X)/24$ of spacetime-filling threebranes for tadpole cancellation [25]. This should be compared with a potentially dual heterotic compactification on an elliptic Calabi-Yau Z with vector bundle V embedded in $E_8 \times E_8$. There the threebranes should correspond to a number n_5 of fivebranes wrapping the elliptic fiber. Their necessity for a consistent heterotic compactification (independent of any duality considerations) was established in the exhaustive study done by Friedman, Morgan and Witten on vector bundles and F-theory [26]. There it was also shown that in the case of an $E_8 \times E_8$ vector bundle V , leaving no unbroken gauge group and corresponding to a smooth Weierstrass model for the fourfold, it is possible to express n_3 and n_5 in comparable and indeed matching data on the common base B .

In this section we show that in dual $N = 1$ string vacua provided by the heterotic string on an elliptic Calabi-Yau together with a $SU(n_1) \times SU(n_2)$ vector bundle respectively F-theory on Calabi-Yau fourfold the number of heterotic fivebranes matches the number of F-theory threebranes. This extends to the general case the work of Friedman, Morgan and Witten, who treated the case of $E_8 \times E_8$ bundle.

Furthermore, it will be presented a complete matching of the number of gauge neutral chiral multiplets obtained in heterotic and F-theory.

5.3.1 Heterotic string on Calabi-Yau threefold

The threefold Z : Let us compactify the heterotic string on a smooth elliptic Calabi-Yau threefold $Z \rightarrow B_2$. We assume that the elliptic fibration has only one section so that $h^{1,1}(Z) = h^{1,1}(B_2) + 1 = c_2(B_2) - 1$. Recall from *section 3*, the Euler characteristic

of Z is given by

$$\chi(Z) = -60 \int_{B_2} c_1^2(B_2) = 2(h^{1,1}(Z) - h^{2,1}(Z)) \quad (5.39)$$

Using Noethers formula $1 = \frac{c_1^2(B_2) + c_2(B_2)}{12}$, we can write the number of complex and Kähler deformations as

$$h^{1,1}(Z) = 11 - c_1^2(B_2), \quad h^{2,1}(Z) = 11 + 29c_1^2(B_2). \quad (5.40)$$

The bundle V : In addition to Z we have to specify a vector bundle V over Z . We are interested in $V = E_8 \times E_8$ and $SU(n_1) \times SU(n_2)$ in order to break the gauge group completely, respectively, to some subgroup of $E_8 \times E_8$. In what follows, we consider vector bundles V which are τ -invariant, i.e. with no additional 'twists' ($\pi_*(\gamma^2) = 0$). Let us collect the necessary bundle data for $V = V_1 \times V_2$:

$E_8 \times E_8$: We find for the number of bundle moduli

$$h^1(Z, \text{ad}(V)) = 16 - 4((\lambda(V_1) + \lambda(V_2)) - (\eta_1 + \eta_2)\sigma) + (\eta_1 + \eta_2)c_1 + 2n_0 \quad (5.41)$$

and the fundamental characteristic class is given by

$$\lambda(V_1) + \lambda(V_2) = (\eta_1 + \eta_2)\sigma - 15(\eta_1^2 + \eta_2^2) + 135(\eta_1 + \eta_2)c_1 - 620c_1^2. \quad (5.42)$$

$SU(n_1) \times SU(n_2)$: We find for the number of bundle moduli

$$h^1(Z, \text{ad}(V)) = rk - 4((c_2(V_1) + c_2(V_2)) - (\eta_1 + \eta_2)\sigma) + (\eta_1 + \eta_2)c_1 + 2n_0 \quad (5.43)$$

with $rk = n_1 + n_2 - 2$ and the second Chern class

$$c_2(V_1) + c_2(V_2) = (\eta_1 + \eta_2)\sigma + (\omega_1 + \omega_2) \quad (5.44)$$

The anomaly: We already mentioned in *section 2.2* that the perturbative anomaly cancellation condition $c_2(V) = c_2(Z)$ will be modified, due to the occurrence of non-perturbative fivebranes in the vacuum. We have

$$E_8 : \lambda(V_1) + \lambda(V_2) + [W] = c_2(TZ) \quad (5.45)$$

$$SU(n) : c_2(V_1) + c_2(V_2) + [W] = c_2(TZ) \quad (5.46)$$

where $[W]$ denotes the cohomology class of the five-branes.

The moduli: Now, we can determine the number of moduli, which lead to a number C_{het} of $N = 1$ neutral chiral (resp. anti-chiral) multiplets

$$C_{het} = h^{2,1}(Z) + h^{1,1}(Z) + h^1(Z, \text{ad}(V)). \quad (5.47)$$

Remark: In general, heterotic string compactifications involve a number of $N = 1$ chiral matter multiplets, which are charged under the unbroken gauge group. This number is given by (c.f. *section 3.2*), using $N_{gen} = \frac{1}{2} | \int_Z c_3(V) |$ we get

$$C_{het}^c = N_{gen} + 2\bar{n}_{gen}. \quad (5.48)$$

Now let us turn to the F-theory side!

5.3.2 F-Theory On Calabi-Yau fourfold

Let us compute for a *general* Calabi-Yau fourfold X_4 the spectrum of massless $N = 1$ superfields in the F-theory compactification, from the Hodge numbers of X_4 . Just as in the six-dimensional case the Hodge numbers of the type IIB bases B_3 , i.e. the details of the elliptic fibrations, will enter the numbers of massless fields. So consider first the compactification of the type IIB string from ten to four dimensions on the spaces B_3 . Abelian $U(1)$ $N = 1$ vector multiplets arise from the dimensional reduction of the four-form antisymmetric Ramond-Ramond tensor field A_{MNPQ} in ten dimensions; therefore we expect that the rank of the four-dimensional gauge group, $r(V)$, gets contributions from the $(2, 1)$ -forms of B_3 such that $h^{(2,1)}(B_3)$ contributes to $r(V)$. Chiral (respectively anti-chiral) $N = 1$ multiplets, which are uncharged under the gauge group, arise from A_{MNPQ} with two internal Lorentz indices as well from the two two-form fields $A_{MN}^{1,2}$ (Ramond-Ramond and NS-NS) with zero or two internal Lorentz indices. Therefore we expect that the number of singlet chiral fields, C , receives contributions from $h^{(1,1)}(B_3)$. On the other hand we can study the F-theory spectrum in three dimensions upon further compactification on a circle S^1 . This is equivalent to the compactification of 11-dimensional supergravity on the same X_4 . (Equivalently we could also consider the compactifications of the IIA superstring on X_4 to two dimensions.) So in three dimensions the 11-dimensional three-form field A_{MNP} contributes $h^{(1,1)}(X_4)$ to $r(V)$ and $h^{(2,1)}$ to C . In addition, the complex structure deformations of the 11-dimensional metric contributes $h^{(3,1)}(X_4)$ chiral fields. (These fields arise in analogy to the $h^{(2,1)}$

complex scalars which describe the complex structure of the metric when compactifying on a Calabi-Yau threefold from ten to four dimensions.) Since, however, vector and chiral fields are equivalent in three dimensions by Poincare duality, this implies that the sum $r(V) + C$ must be independent from the Hodge numbers of the type IIB bases B^3 . Therefore, just in analogy to the six-dimensional F-theory compactifications, the following formulas for the spectrum of the four-dimensional F-theory models on Calabi-Yau fourfolds are expected [[30],[79],[49]]: namely for the rank of the $N = 1$ gauge group we derive

$$r(V) = h^{(1,1)}(X_4) - h^{(1,1)}(B_3) - 1 + h^{(2,1)}(B_3) \quad (5.49)$$

and for the number C_F of $N = 1$ neutral chiral (resp. anti-chiral) multiplets we get

$$\begin{aligned} C_F &= h^{(1,1)}(B_3) - 1 + h^{(2,1)}(X_4) - h^{(2,1)}(B_3) + h^{(3,1)}(X_4) \\ &= h^{(1,1)}(X_4) - 2 + h^{(2,1)}(X_4) + h^{(3,1)}(X_4) - r(V) \\ &= \frac{\chi}{6} - 10 + 2h^{(2,1)}(X_4) - r(V). \end{aligned} \quad (5.50)$$

Note that in this formula we did not count the chiral field which corresponds to the dual heterotic dilaton. Further to get a consistent F-theory compactification on X_4 we have to include a number n_3 of space-time filling threebranes in the vacuum [25].

$$n_3 = \frac{\chi(X_4)}{24} \quad (5.51)$$

Note: We do not turn on gauge bundles inside the sevenbrane, moreover, we assume that no fourflux is turned on (since the fourflux in F-theory is related to the 'twists' of the line bundle on the spectral cover: $\pi_*(\gamma^2) = -G^2$ [61].)

5.3.3 The duality

To establish the four-dimensional het/F-theory duality we will implement the adiabatically extended duality by the following specification [26]: as X_4 is assumed to be a $K3$ fibration over B_2 it follows that B_3 , the threefold base of the F-theory elliptic fibration, is a \mathbf{P}^1 fibration over B_2 ; this fibration structure is described by assuming the \mathbf{P}^1 bundle over B_2 to be a projectivization of a vector bundle $Y = \mathcal{O} \oplus \mathcal{T}$, with \mathcal{T} a line bundle over B_2 (c.f. *section 3*); then the cohomology class $t = c_1(\mathcal{T})$ encodes the \mathbf{P}^1 fibration structure. Now the duality is implemented by choosing for our bundle

$$\eta_1 = 6c_1(B_2) + t, \quad \eta_2 = 6c_1(B_2) - t \quad (5.52)$$

where $\eta = \pi_*(\lambda(V))$ in the case of $V = E_8$ and $\eta = \pi_*(c_2(V))$ for $V = SU(n)$. Note that this specification is the analogue of the relation in the 6D het/F- theory duality [21] with $(12 + n, 12 - n)$ instantons embedded in each E_8 and F-theory on elliptic fibered threefold over F_n .

Brane Match:

Let us first recall to what was done by Friedman, Morgan and Witten. They considered $E_8 \times E_8$ bundle and a smooth Calabi-Yau fourfold. With $\eta_{1,2} = 6c_1(B_2) \pm t$ one can express the fundamental characteristic class of the E_8 bundle as

$$\lambda(V_1) + \lambda(V_2) = -80c_1^2 + 12\sigma c_1 - 30t^2 \quad (5.53)$$

and using further $c_2(Z) = c_2 + 11c_1^2 + 12\sigma c_1$ one finds the number of fivebranes [26]

$$n_5 = c_2 + 91c_1^2 + 30t^2 \quad (5.54)$$

On the F-theory side, one has to express $\chi(X_4)$ in terms of base B_2 data and finds for the number of threebranes

$$n_3 = \int_{B_2} c_2 + 91c_1^2 + 30t^2. \quad (5.55)$$

which matches the heterotic fivebranes.

Now let us see how this matching proceeds in the case of $V = SU(n_1) \times SU(n_2)$. To achieve this we will adopt a somewhat different technical procedure [80]. In the case of not having a smooth Weierstrass model there occurred already in [25] the difficulty to reduce the fourfold expression $\chi(X_4)/24$ for n_3 to an expression involving only suitable data of the base B_3 of the elliptic F-theory fibration of X_4 (not to be confused with the twofold base B_2 of the $K3$ fibration of X_4 , here denoted simply by B_2 , which is visible also on the heterotic side), which was [26] only an intermediate step to reduce the expression to one involving only twofold base data. For this reason we express here $\chi(X)$ directly in the Hodge numbers of X_4 and match then these with the data of the dual heterotic model; here essential use is made of the index-formula computation.¹³

Let us start and first compute the number of heterotic five-branes n_5 using the nonperturbative anomaly cancellation condition $n_5 = c_2(Z) - (c_2(V_1) + c_2(V_2))$. Here

¹³Note that in case some of the threebranes have dissolved into finite-sized instantons in the world-volume gauge theory of the F-theory seven-brane, this is accompanied by a corresponding five-brane transition on the heterotic side [62], [26], [69].

we have to recall that this formula was derived [26] under the assumption that Z admits a smooth Weierstrass model, an assumption we adopt here as already stated (note that in contrast to the F-theory side, where this bears essential physical content, it represents on the heterotic side just a technical assumption). For explicit evaluation and to establish some notation we give here the actual number of fivebranes for V a $SU(n_1) \times SU(n_2)$ bundle. The second Chern class for a $SU(n)$ bundle was given by (with $\eta = \pi_*(c_2(V))$ and \mathcal{L} being some line bundle over B_2)

$$\begin{aligned} c_2(V) &= \eta\sigma + \omega \\ &= \eta\sigma - \frac{1}{24}c_1(\mathcal{L})^2(n^3 - n) - \frac{n}{8}\eta(\eta - nc_1(\mathcal{L})) \end{aligned} \quad (5.56)$$

Using the above relations for η_1 , η_2 and $c_2(V)$ we can derive (where also $c_1(B) = c_1(\mathcal{L})$ by the Calabi-Yau condition for Z)

$$c_2(V_{1/2}) = 6c_1\sigma \pm t\sigma - \frac{1}{24}c_1^2(n_1^3 - n_1) - \frac{n_1}{8}[36c_1^2 \pm 12c_1 t + t^2 - 6n_1c_1^2 \mp n_1 t c_1] \quad (5.57)$$

We find for the number of fivebranes

$$\begin{aligned} n_5 &= c_2 + 11c_1^2 + \frac{1}{24}c_1^2(n_1^3 - n_1 + n_2^3 - n_2) + \frac{(n_1 + n_2)}{4}(18c_1^2 + \frac{t^2}{2}) + \\ &\quad + \frac{(n_1 - n_2)}{4}6c_1 t - \frac{(n_1^2 + n_2^2)}{8}6c_1^2 + \frac{(n_2^2 - n_1^2)}{8}t c_1 \end{aligned} \quad (5.58)$$

We can now express the number of F-theory threebranes $n_3 = \chi(X_4)/24$ in terms of heterotic data. Because of $\chi(X)/6 - 8 = h^{1,1}(X_4) + h^{3,1}(X_4) - h^{2,1}(X_4)$ we have to use the following informations ¹⁴

$$h^{1,1}(X_4) = h^{1,1}(Z) + 1 + r = 12 - c_1^2 + r \quad (5.59)$$

$$h^{2,1}(X_4) = n_o \quad (5.60)$$

$$h^{3,1}(X_4) = h^{2,1}(Z) + I + n_o + 1 = 12 + 29c_1^2 + I + h^{2,1}(X) \quad (5.61)$$

where in the first line we used Noethers formula and $\chi(Z) = -60 \int_{B_2} c_1^2(B_2)$; for the n_o in the second line see below. Let us now see how the expression for $h^{3,1}(X_4)$ emerges. Note that the moduli space \mathcal{M} for bundles on Z of dimension m_{bun} has a fibration $\mathcal{M} \rightarrow \mathcal{Y}$ which corresponds on the F-theory side to a fibration of the abelian variety

¹⁴ r denotes the rank of the unbroken non-abelian gauge group (do not confuse it with the rank of the group of the bundle V); we furthermore assume that we have no further $U(1)$ factors (coming from sections).

$H^3(X_4, \mathbf{R})/H^3(X_4, \mathbf{Z})$ of complex dimension $h^{2,1}(X_4)$ over a part of the space of complex deformations of X_4 (cf. [26]); the remaining complex deformations account for the complex deformations of the heterotic Calabi-Yau (+1), i.e.(see below)

$$h^{3,1}(X_4) + h^{2,1}(X_4) = h^{2,1}(Z) + m_{bun} + 1$$

Now concerning the contribution of the bundle moduli recall I is given by $I = -\sum_{i=o}^3 (-1)^i h^i(Z, \text{ad}(V))_e$ where the subscript "e" (resp. "o") indicates the even (resp. odd) part. As we have an unbroken gauge group H , which is the commutator of the group G of V , one finds $I = n_e - n_o$ corrected by $h_e^0 - h_o^0$ denoting by $n_{e/o}$ the number $h^1(Z, \text{ad}(V))_{e/o}$ of massless even/odd chiral superfields and by $h_{e/o}^0$ the number of unbroken gauge group generators even/odd under τ . The unbroken gauge group is in this n_3 - calculation accounted for by the rank contribution in $h^{1,1}(X_4)$ for the resolved fourfold.

So one has for the number of bundle moduli $m_{bun} = h^1(Z, \text{ad}(V)) = n_e + n_o = I + 2n_o$ and so gets the announced expression for $h^{3,1}(X_4)$. Furthermore on the F-theory side the modes odd under the involution τ' corresponding to the heterotic involution τ correspond to the $h^{2,1}(X_4)$ classes [26] (we assume no four-flux was turned on). Using the formula for the index of $SU(n_1) \times SU(n_2)$ bundle and the expressions for η_1 and η_2 and $c_2(Z)$ and denoting $rk = n_1 + n_2 - 2 = 16 - r$ we find for I

$$\begin{aligned} I &= rk - 4(c_2(V_1) + c_2(V_2)) + 48c_1\sigma + 12c_1^2 \\ &= rk - 4(c_2 + 11c_1^2 + 12\sigma c_1) + 4n_5 + 48c_1\sigma + 12c_1^2 \\ &= rk - (48 + 28c_1^2) + 4n_5 \end{aligned} \tag{5.62}$$

One finally gets then with $r = 16 - rk$ and our expression for I that

$$\begin{aligned} \chi(X_4)/24 &= 2 + \frac{1}{4}(12 - c_1^2 + 16 - rk + 12 + 29c_1^2 + I) \\ &= 2 + \frac{1}{4}(40 + 28c_1^2 + I - rk) \\ &= n_5 \end{aligned} \tag{5.63}$$

matching the heterotic value.

After showing that the number of heterotic fivebranes matches the number of F-theory threebranes for E_8 and $SU(n)$ vector bundle on Z , we now turn to the moduli match.

Moduli Match:

Let us now compare the number of moduli in heterotic and F-theory which lead to $N = 1$ neutral chiral multiplets. Recall, this number was given by

$$C_{het} = h^{2,1}(Z) + h^{1,1}(Z) + h^1(Z, \text{ad}(V)) \quad (5.64)$$

$$= 22 + 28c_1^2 + I + 2n_0 \quad (5.65)$$

using $h^{1,1}(Z) = 11 - c_1^2(B_2)$ resp. $h^{2,1}(Z) = 11 + 29c_1^2(B_2)$.

On the F-theory side we have

$$C_F = \frac{\chi(X_4)}{6} - 10 + 2h^{(2,1)}(X^4) - r(V). \quad (5.66)$$

using $\chi(X_4) = 6(8 + h^{1,1}(X_4) + h^{2,1}(X_4) + h^{3,1}(X_4) - h^{2,1}(X_4))$ and (5.59-5.61) for the hodge numbers, we get

$$C_F = 22 + 28c_1^2 + I + 2h^{2,1}(X_4) \quad (5.67)$$

So, if we identify the modes which are odd under the τ' involution on the F-theory side with those odd under τ -involution on the heterotic side, i.e.

$$n_0 = h^{2,1}(X_4). \quad (5.68)$$

we get complete matching.

Note that this is possible as long as we assume that no four-flux is turned on which otherwise would imply that we have to take into account the twistings appearing in the spectral cover construction of V . The twistings lead to a multi-component structure of the bundle moduli space (c.f.[61],[62])!

Let us consider an example! The case $V = E_8 \times E_8$ on Z which means on the F-theory side that there is no unbroken gauge group!

The number of geometrical and bundle moduli which leads to the number C_{het} of $N = 1$ neutral chiral multiplets can be written as

$$\begin{aligned} C_{het} &= h^{2,1}(Z) + h^{1,1}(Z) + h^1(Z, \text{ad}(V)) \\ &= 38 + 360 \int_{B_2} c_1^2(B_2) + 120 \int_{B_2} t^2 + 2n_0 \end{aligned} \quad (5.69)$$

using $h^{2,1} + h^{1,1} = 22 + 28 \int_{B_2} c_1^2(B_2)$ and the number of bundle moduli $h^1(Z, \text{ad}(V)) = I + 2n_0$ which we express in terms of $\eta_{1,2} = 6c_1(B_2) \pm t$, one has (c.f.[49])

$$I = 16 + 332 \int_{B_2} c_1^2 + 120 \int_{B_2} t^2. \quad (5.70)$$

The expected number C_F of F-theory neutral chiral multiplets is given by

$$C_F = \frac{\chi(X_4)}{6} - 10 + 2h^{(2,1)}(X_4) - r(V). \quad (5.71)$$

Since we have no unbroken gauge group we have $r(V) = 0$ and if we use

$$\frac{\chi(X_4)}{24} = 12 + 90 \int_{B_2} c_1^2(B_2) + 30 \int_{B_2} t^2 \quad (5.72)$$

we will find

$$C_F = 38 + 360 \int_{B_2} c_1^2(B_2) + 120 \int_{B_2} t^2 + 2h^{2,1}(X_4) \quad (5.73)$$

and thus getting complete matching (identifying $n_0 = h^{2,1}(X_4)$)

$$C_{het} = C_F \quad (5.74)$$

6 4D Het/F-Models

In the following, we consider $N = 1$ dual F-theory/heterotic string pairs with B_2 being the so-called del Pezzo surfaces dP_k , the \mathbf{P}^2 's blown up in k points. Then, in our class of models the three-dimensional bases, $B_{n,k}^3$, are characterized by two parameters k and n , where n encodes the fibration structure $B_{n,k}^3 \rightarrow dP_k$. Specifically we will discuss the cases $n = 0$, $k = 0, \dots, 6$, i.e. $B_{0,k}^3 = \mathbf{P}^1 \times dP_k$, and also the cases $n = 0, 2, 4, 6, 12$, $k = 9$, where the 9 points are the intersection of two cubics and the $B_{n,9}$ are a fibre product $dP_9 \times_{\mathbf{P}^1} F_{n/2}$ with common base \mathbf{P}^1 and $F_{n/2}$ the rational ruled Hirzebruch surfaces. We will show that the massless spectra match on both sides.

The considered class of models is the direct generalization of the F-theory/heterotic $N = 1$ dual string pair with $n = 0$, $k = 9$ (again the 9 points in the mentioned special position), which were investigated in [30], which was a \mathbf{Z}_2 modding of a $N = 2$ model described as F-theory on $T^2 \times CY^3$ (also considered there was the \mathbf{Z}_2 modding of $K3 \times K3$). In this model it was also considered a duality check for the superpotential.

The first chain of models we will discuss here consists then in heterotic Calabi-Yau's elliptic over dP_k ($k = 0, \dots, 3$) with a *general* $E_8 \times E_8$ vector bundle resp. F-theory on smooth Calabi-Yau fourfolds X_k^4 elliptic over $dP_k \times \mathbf{P}^1$. The heterotic Calabi-Yau of $\chi \neq 0$ varies with k and one can also see the transition induced from the blow-ups in \mathbf{P}^2 among the fourfolds on the F-theory side (for transitions among $N = 1$ vacua cf. also [81]).

Then we go on to construct a second chain of four-dimensional F-theory/heterotic dual string pairs with $N = 1$ supersymmetry by \mathbf{Z}_2 -modding of corresponding dual pairs with $N = 2$ supersymmetry. The resulting Calabi-Yau four-folds X_n^4 are $K3$ -fibrations over the del Pezzo surface dP_9 (with points in special positions). On the heterotic side, the dual models are obtained by compactification on a known Voisin-Borcea Calabi-Yau three-fold with Hodge numbers $h^{(1,1)} = h^{(2,1)} = 19$, where, similarly as in the underlying $N = 2$ models, the heterotic gauge bundles over this space are characterized by turning on $(6 + \frac{n}{2}, 6 - \frac{n}{2})$ instantons of $E_8 \times E_8$. We work out the Higgsing chains of the gauge groups together with their massless matter content (for example the numbers of chiral multiplets in the $\mathbf{27} + \overline{\mathbf{27}}$ representation of E_6) for each model and show that the heterotic spectra of our models match the dual F-theory spectra, as computed from the Hodge numbers of the four-folds X_n^4 and of the type IIB base spaces.

Note that in contrast to this second class of models the first chain (varying k , $n = 0$)

consists of *genuine* $N = 1$ models. Moreover, within the first chain the heterotic Calabi-Yau spaces have non-vanishing Euler numbers, potentially leading to theories with chiral spectra with respect to non-Abelian gauge groups which show up at certain points in the moduli spaces. Hence the transitions in k might connect $N = 1$ models with different numbers of chiral fermions.

6.1 F-theory on $X_k^4 \rightarrow \mathbf{P}^1 \times dP_k$

Let us start and consider F-theory on a smooth elliptically fibered Calabi-Yau fourfold [49] X^4 with base $B_k^3 = \mathbf{P}^1 \times dP_k$ which can be represented by a smooth Weierstrass model over B_k^3 if the anti-canonical line bundle $-K_B$ over B_k^3 is very ample [82]. The Weierstrass model is can be described as above by the homogeneous equation $y^2z = x^3 + g_2xz^2 + g_3z^3$ in a \mathbf{P}^2 bundle $W \rightarrow B_k^3$. The \mathbf{P}^2 bundle is the projectivization of a vector bundle $K_B^{-2} \oplus K_B^{-3} \oplus \mathcal{O}$ over B_k^3 .

Most of the 84 known Fano threefolds have a very ample $-K_B$ [82]. 18 of them are toric threefolds \mathcal{F}_n ($1 \leq n \leq 18$) and are completely classified [83], [84], [85], [86]. They were studied in the context of Calabi-Yau fourfold compactifications over Fano threefolds [44], [79]. In particular in [82], it was shown that for $k = 0, \dots, 6$ over B_k^3 there exists a smooth Weierstrass model (having a section); B_k^3 with $k = 0, 1, 2, 3$ correspond to the toric Fano threefolds \mathcal{F}_n with $n = 2, 9, 13, 17$. The corresponding fourfolds X^4 have a $K3$ fibration over dP_k (cf. [79],[87]). First we can determine the Hodge numbers of B_k^3 . The number of Kähler and complex structure parameters of B_k^3 are $h^{11}(B_k^3) = 2 + k$ (where the 2 is comes from the line in \mathbf{P}^2 and the class of the \mathbf{P}^1) and $h^{21}(B_k^3) = 0$. For $k = 0, \dots, 6$ we can compute the Euler number of X in terms of topological data of the base

$$\chi(X_k^4) = 12 \int_{B_k^3} c_1 c_2 + 360 \int_{B_k^3} c_1^3 \quad (6.1)$$

where the c_i refer to B_k^3 (the 360 is related to the Coxeter number of E_8 which is associated with the elliptic fiber type [44]).

One finds $\int_{B_k^3} c_1 c_2 = 24$ and $\int_{B_k^3} c_1^3 = 3c_1(\mathbf{P}^1)c_1(dP_k)^2 = 6(9 - k)$ which leads to

$$\chi(X_k^4) = 288 + 2160(9 - k). \quad (6.2)$$

In the following we restrict to $k = 0, 1, 2, 3$ where one has $h^{21} = 0$, [44],[79], $h^{11}(X_k^4) = 3 + k$, $h^{31}(X_k^4) = \frac{\chi}{6} - 8 - (3 + k) = 28 + 361(9 - k)$ [44],[79],[25]. Now since we have a

smooth Weierstrass model and thus expecting no unbroken gauge group, i.e.

$$\begin{aligned} r(V) &= h^{11}(X_k^4) - h^{11}(B_k^3) - 1 + h^{21}(B_k^3) \\ &= 0 \end{aligned} \tag{6.3}$$

and for the number of N=1 neutral chiral (resp. anti-chiral) multiplets C_F we get

$$\begin{aligned} C_F &= h^{11}(B_k^3) - 1 + h^{21}(X_k^4) - h^{21}(B_k^3) + h^{31}(X_k^4) \\ &= 38 + 360(9 - k) \end{aligned} \tag{6.4}$$

k	χ	h^{31}	h^{22}	h^{11}	C_F
0	19728	3277	13164	3	3278
1	17568	2916	11724	4	2918
2	15408	2555	10284	5	2558
3	13248	2194	8844	6	2198
4	11088	1833	7404	7	1838
5	8928	1472	5964	8	1478
6	6768	1111	4524	9	1118

In this table we have made for the cases $k = 4, 5, 6$ the assumption $h^{21}(X^4) = 0$. With this assumption, the matching we will present in the following goes through also in these cases.

Furthermore, for the number of threebranes n_3 we have

$$n_3 = \int_{B^2} (c_2 + 91c_1^2 + 30t^2) = 822 - 90k \tag{6.5}$$

using $t = 0$ since we have $B_k^3 = \mathbf{P}^1 \times dP_k$ (recall t measures the non-triviality of the \mathbf{P}^1 bundle over $B_2 = dP_k$.)

From the last expression we learn that between each blow up there is a threebrane difference of 90. Note that Sethi, Vafa and Witten [25] had a brane difference of 120 as they blow up in a threefold whereas we do this in B^2 .

Now let us turn to the heterotic side!

6.2 Heterotic string on $Z \rightarrow dP_k$

To compactify the heterotic string on Z , we have in addition to specify our vector bundle V with fixed second Chern class. Since we had zero for the rank $r(V)$ of the $N=1$ gauge group on the F-theory side we have to switch on a $E_8 \times E_8$ bundle that breaks the gauge group completely.

Now let Z be a nonsingular elliptically fibered Calabi-Yau threefold over dP_k with a section. Recall the Picard group of dP_k is $\text{Pic } dP_k = \mathbf{Z}\ell \oplus \mathbf{Z}\ell_i$ where ℓ denotes the class of the line in \mathbf{P}^2 and ℓ_i , $i = 1, \dots, k$ are the classes of the blown up points. The intersection form is defined by [88]

$$\ell^2 = 1, \quad \ell_i \cdot \ell = 0, \quad \ell_i \cdot \ell_j = -\delta_{ij}, \quad (6.6)$$

and the canonical class of dP_k is

$$\mathcal{K}_B = -3\ell + \sum_{i=1}^k \ell_i. \quad (6.7)$$

Assuming again that we are in the case of a general¹⁵ smooth Weierstrass model one has $h^{11}(Z) = 2 + k$. So the fourth homology of Z is generated by the following divisor classes: $D_0 = \pi^*\ell$, $D_i = \pi^*\ell_i$ and $S = \sigma(\mathcal{B}_k)$. The intersection form on Z is then given by

$$S^3 = 9 - k, \quad D_0^2 S = 1, \quad D_i^2 S = -1, \quad D_i S^2 = -1, \quad D_0 S^2 = -3, \quad (6.8)$$

all other triple intersections are equal to zero.

The canonical bundle for $Z \rightarrow B$ is given by [22],[23]

$$\mathcal{K}_Z = \pi^*(\mathcal{K}_B + \sum a_i[\Sigma_i]) \quad (6.9)$$

where a_i are determined by the type of singular fiber and Σ_i is the component of the locus within the base on which the elliptic curve degenerates. In order to get \mathcal{K}_Z trivial, one requires $\mathcal{K}_B = -\sum a_i[\Sigma_i]$.

Since we are interested in a smooth elliptic fibration, we have to check that the elliptic fibration does not degenerate worse than with I_1 singular fiber over codimension one

¹⁵i.e. having only one section [82], so a typical counterexample would be the $CY^{19,19} = B_9 \times_{\mathbf{P}^1} B_9$ with the 9 points being the intersection of two cubics

in the base which then admits a smooth Weierstrass model. That this indeed happens in our case of del Pezzo base can be proved by similar methods as for the F_n case in [22]. As $a_i = -1$ we have a smooth elliptic fibration.

The Euler number of Z is given by (as above) $\chi(Z) = -60 \int c_1^2(B)$ and¹⁶

$$h^{11}(Z) = 2 + k = 11 - (9 - k) \quad (6.10)$$

$$h^{21}(Z) = (272 - 29k) = 11 + 29(9 - k). \quad (6.11)$$

With the Index of the $E_8 \times E_8$ vector bundle

$$I = 16 + 332 \int_B c_1^2(B) + 120 \int_B t^2 \quad (6.12)$$

recalling that in our case the last term vanishes, we find for the number of bundle moduli (with $n_0 = 0$ since we have $h^{2,1}(X^4) = 0$)

$$I = 16 + 332(9 - k) \quad (6.13)$$

and for the number of $N = 1$ neutral chiral (resp. antichiral) multiplets C_{het} we find

$$\begin{aligned} C_{het} &= h^{21}(Z) + h^{11}(Z) + I \\ &= 38 + 360(9 - k) \end{aligned} \quad (6.14)$$

which agrees with the number of chiral multiplets C_F on the F-theory side.

k	χ	h^{21}	h^{11}	I
0	-540	272	2	3004
1	-480	243	3	2672
2	-420	214	4	2340
3	-360	185	5	2008
4	-300	156	6	1676
5	-240	127	7	1344
6	-180	98	8	1012

¹⁶Note that the transition by 29 in $h^{21}(Z)$ in going from k to $k + 1$ has a well known interpretation if one uses these Calabi-Yau threefolds as F-theory compactification spaces cf. [89]

6.3 F-theory on the $K3$ -fibred fourfolds X_n^4

6.3.1 The $N = 2$ models: F-theory on $X_n^3(\times T^2)$

We will start constructing the fourfolds X_n^4 by first considering F-theory compactified to six dimensions on an elliptic Calabi-Yau threefold X^3 , and then further to four dimensions on a two torus T^2 , i.e. the total space is given by $X^3 \times T^2$. This leads to $N = 2$ supersymmetry in four dimensions. As explained in [21], this four-dimensional F-theory compactification is equivalent to the type IIA string compactified on the same Calabi-Yau X^3 .

To be more specific, let us discuss the cases where the Calabi-Yau threefolds, which we call X_n^3 , are elliptic fibrations over the rational ruled Hirzebruch surfaces F_n as already mentioned in section 5. The corresponding type IIB base spaces are given by the Hirzebruch surfaces F_n in six dimensions. X_n^3 is given by (c.f. section 5)

$$X_n^3: \quad y^2 = x^3 + \sum_{k=-4}^4 f_{8-nk}(z_1) z_2^{4-k} x + \sum_{k=-6}^6 g_{12-nk}(z_1) z_2^{6-k}. \quad (6.15)$$

with $f_{8-nk}(z_1)$, $g_{12-nk}(z_1)$ are polynomials of degree $8 - nk$, $12 - nk$ respectively, where the polynomials with negative degrees are identically set to zero. From this equation we see that the Calabi-Yau threefolds X_n^3 are $K3$ fibrations over $P_{z_1}^1$ with coordinate z_1 ; the $K3$ fibres themselves are elliptic fibrations over the $P_{z_2}^1$ with coordinate z_2 .

Recall, $h^{(2,1)}(X_n^3)$, counting the number of complex structure deformations of X_n^3 , are given by the the number of parameters of the curve (6.15) minus the number of possible reparametrizations, which are given by 7 for $n = 0, 2$ and by $n + 6$ for $n > 2$. On the other hand, the Hodge numbers $h^{(1,1)}(X_n^3)$, which count the number of Kähler parameters of X_n^3 , are determined by the Picard number ρ of the $K3$ -fibre of X_n^3 as

$$h^{(1,1)}(X_n^3) = 1 + \rho. \quad (6.16)$$

Let us list the Hodge numbers of the X_n^3 for those cases relevant for our following discussion:

n	$h^{(1,1)}(X_n^3)$	$h^{(2,1)}(X_n^3)$
0	3	243
2	3	243
4	7	271
6	9	321
12	11	491

(6.17)

Further recall, the Hodge numbers of X_n^3 determine the spectrum of the F-theory compactifications. In six dimensions the number of tensor multiplets T was given

$$n_T = h^{(1,1)}(B^2) - 1. \quad (6.18)$$

The rank of the six-dimensional gauge group is given by

$$r(V) = h^{(1,1)}(X_n^3) - h^{(1,1)}(B^2) - 1. \quad (6.19)$$

Finally, the number of hypermultiplets n_H , which are neutral under the Abelian gauge group is

$$n_H = h^{(2,1)}(X_n^3) + 1. \quad (6.20)$$

For the cases we are interested in, namely $B^2 = F_n$, $h^{(1,1)}(F_n)$ is universally given by $h^{(1,1)}(F_n) = 2$. Therefore one immediately gets $n_T = 1$, which corresponds to the universal dilaton tensor multiplet in six dimensions, and

$$r(V) = \rho - 2 = h^{(1,1)}(X_n^3) - 3. \quad (6.21)$$

At special loci in the moduli spaces of the hypermultiplets one obtains enhanced non-Abelian gauge symmetries. These loci are determined by the singularities of the curve (6.15) and were analyzed in detail in [76]. These F-theory singularities correspond to the perturbative gauge symmetry enhancement in the dual heterotic models.

6.3.2 The $N = 1$ models: F-theory on $X_n^4 = (X_n^3 \times T^2)/\mathbf{Z}_2$

Now we will construct from the $N = 2$ F-theories on $X_n^3 \times T^2$ the corresponding $N = 1$ models on Calabi-Yau four-folds X_n^4 by a \mathbf{Z}_2 modding procedure, i.e.

$$X_n^4 = \frac{X_n^3 \times T^2}{\mathbf{Z}_2}. \quad (6.22)$$

First, the \mathbf{Z}_2 modding acts as quadratic redefinition on the coordinate z_1 , the coordinate of the base $\mathbf{P}_{z_1}^1$ of the $K3$ -fibred space X_n^3 , i.e. the operation is $z_1 \rightarrow -z_1$. This means that the modding is induced from the quadratic base map $z_1 \rightarrow \tilde{z}_1 := z_1^2$ with the two branch points 0 and ∞ . So the degrees of the corresponding polynomials $f(z_1)$ and $g(z_1)$ in eq.(6.15) are reduced by half (i.e. the moddable cases are the ones where only even degrees occur). So instead of the Calabi-Yau threefolds X_n^3 , we are now dealing

with the non-Calabi-Yau threefolds $\mathcal{B}_n^3 = X_n^3/\mathbf{Z}_2$ which can be written in Weierstrass form as follows:

$$\mathcal{B}_n^3 : \quad y^2 = x^3 + \sum_{k=-4}^4 f_{4-\frac{nk}{2}}(z_1) z_2^{4-k} x + \sum_{k=-6}^6 g_{6-\frac{nk}{6}}(z_1) z_2^{6-k}. \quad (6.23)$$

The \mathcal{B}_n^3 are now elliptic fibrations over $F_{n/2}$ and still $K3$ fibrations over $\mathbf{P}_{z_1}^1$. Note that the unmodded 3-folds X_n^3 and the modded spaces \mathcal{B}_n^3 have still the same $K3$ -fibres with Picard number ρ . The Euler numbers of \mathcal{B}_n^3 can be computed from the Euler numbers of X_n^3 from the ramified covering as

$$\chi(X_n^3) = 2\chi(\mathcal{B}_n^3) - 2 \cdot 24. \quad (6.24)$$

Using $\chi(X_n^3) = 2(1 + \rho - h^{(2,1)}(X_n^3))$ and $\chi(\mathcal{B}_n^3) = 2 + 2(1 + \rho - h^{(2,1)}(\mathcal{B}_n^3))$ we derive the followong relation between $h^{(2,1)}(X_n^3)$ and $h^{(2,1)}(\mathcal{B}_n^3)$:

$$h^{(2,1)}(\mathcal{B}_n^3) = \frac{1}{2}(\rho - 2 + h^{(2,1)}(X_n^3) - 19). \quad (6.25)$$

Second, X_n^4 are of course no more products $\mathcal{B}_n^3 \times T^2$ but the torus $T_{z_4}^2$ now is a second elliptic fibre which varies over $\mathbf{P}_{z_1}^1$. More precisely, this elliptic fibration just describes the emergence of the del Pezzo surface dP_9 , which is given in Weierstrass form as

$$dP_9 : \quad y^2 = x^3 + f_4(z_1)x + g_6(z_1). \quad (6.26)$$

Therefore the spaces X_n^4 have the form of being the following fibre products:

$$X_n^4 = dP_9 \times_{\mathbf{P}_{z_1}^1} \mathcal{B}_n^3. \quad (6.27)$$

All X_n^4 are $K3$ fibrations over the mentioned dP_9 surface. The Euler numbers of all X_n^4 's are given by the value

$$\chi = 12 \cdot 24 = 288. \quad (6.28)$$

The corresponding (complex) three-dimensional IIB base manifolds B_n^3 have the following fibre product structure

$$B_n^3 = dP_9 \times_{\mathbf{P}_{z_1}^1} F_{n/2}. \quad (6.29)$$

For the case already studied in [30] with $n = 0$, B_0^3 is just the product space $dP_9 \times \mathbf{P}_{z_2}^1$.

The fibration structure of X_n^4 provides all necessary information to compute the Hodge numbers of X_n^4 from the number of complex deformations of \mathcal{B}_n^3 , which we call

$N_{\mathcal{B}_n^3}$. These can be calculated from eq.(6.23) and are summarized in table (6.36). Note that in the cases $n > 2$ we have to subtract in the $N = 2$ setup $7 + n - 1 = 6 + n$ reparametrizations, whereas in the $N = 1$ setup only $6 + n/2$ (for $n = 0, 2$ we have to subtract 7 reparametrizations both for $N = 1$ and $N = 2$).

Knowing that the number of complex deformations of dP_9 is eight, as easily be read in eq.(6.26), we obtain for the number of complex structure deformations of X_n^4 the following result:

$$h^{(3,1)}(X_n^4) = 8 + 3 + N_{\mathcal{B}_n^3} = 11 + N_{\mathcal{B}_n^3}. \quad (6.30)$$

Next compute the number of Kähler parameters, $h^{(1,1)}(X_n^4)$, of X_n^4 . Since $h^{(1,1)}(dP_9) = 10$ we obtain the formula

$$h^{(1,1)}(X_n^4) = 10 + \rho, \quad (6.31)$$

where ρ is the Picard number of the $K3$ fibre of X_n^4 .

Finally, for the computation of $h^{(2,1)}(X_n^4)$ of X_n^4 we can use the condition [25] of tadpole cancellation, which tells us that $h^{(1,1)}(X_n^4) - h^{(2,1)}(X_n^4) + h^{(3,1)}(X_n^4) = \frac{\chi}{6} - 8$. Hence we get for $h^{(2,1)}(X_n^4)$

$$h^{(2,1)}(X_n^4) = \rho + N_{\mathcal{B}_n^3} - 19. \quad (6.32)$$

Using eqs.(6.30,6.31,6.32), we have summarized the spectrum of Hodge numbers of X_n^4 in table (6.36). The computation of these 4-fold Hodge numbers, which was based on the counting of complex deformations of the Weierstrass form eq.(6.23), can be checked in a rather independent way, by noting that $h^{(2,1)}(X_n^4) = h^{(2,1)}(\mathcal{B}_n^3)$, since $N_{\mathcal{B}_n^3} = (19 - \rho) + h^{(2,1)}(\mathcal{B}_n^3)$. Then, using eq.(6.25) and table (6.17) the Hodge numbers $h^{(2,1)}(X_n^4)$ in table (6.36) are immediately verified.

Let us compute the spectrum C_F and C_{het} for our chain of models using. The Hodge numbers of B_n^3 eq.(6.29) are universally given as $h^{(1,1)}(B_n^3) = 11$, $h^{(2,1)}(B_n^3) = 0$. Thus we obtain using eq.(6.31) that

$$r(V) = h^{(1,1)}(X_n^3) - 12 = \rho - 2; \quad (6.33)$$

observe that the rank of the $N = 1$ four-dimensional gauge groups agrees with the rank of the six-dimensional gauge groups of the corresponding $N = 2$ parent models (see

eq.(6.21)).¹⁷ Second, using eqs.(6.30,6.32) we derive that

$$C_F = 38 - r(V) + 2h^{(2,1)}(X_n^4) = 2 + \rho + 2N_{\mathcal{B}_n^3}. \quad (6.34)$$

Using eq.(6.25), C_F can be expressed by the number of hypermultiplets of the $N = 2$ parent models as follows:

$$C_F = 38 + n_H - 20. \quad (6.35)$$

This relation will become clear when considering the dual heterotic models in the next chapter. The explicit results for $N_{\mathcal{B}_n^3}$, $h^{(1,1)}(X_n^4)$, $h^{(2,1)}(X_n^4)$, $h^{(3,1)}(X_n^4)$, $r(V)$ and C are contained in the following table:

n	$N_{\mathcal{B}_n^3}$	$h^{(1,1)}(X_n^4)$	$h^{(2,1)}(X_n^4)$	$h^{(3,1)}(X_n^4)$	$r(V)$	C
0	129	12	112	140	0	262
2	129	12	112	140	0	262
4	141	16	128	152	4	290
6	165	18	154	176	6	340
12	249	20	240	260	8	510

(6.36)

The equations (6.33) and (6.34) count the numbers of Abelian vector fields and the number of neutral chiral moduli fields of the four-dimensional F-theory compactification. Let us now discuss the emergence of $N = 1$ non-Abelian gauge groups together with their matter contents. Namely, non-Abelian gauge groups arise by constructing 7-branes over which the elliptic fibration has an ADE singularity [32]. Specifically, one has to consider a (complex) two-dimensional space, which is a codimension one subspace of the type IIB base B^3 , over which the elliptic fibration has a singularity. (In order to avoid adjoint matter, the space S must satisfy $h^{(2,0)}(S) = h^{(1,0)}(S) = 0$.) The world volume of the 7-branes is then given by $R^4 \times S$; if n parallel 7-branes coincide, one gets for example an $SU(n)$ gauge symmetry, i.e. the elliptic fibration acquires an A_{n-1} singularity. N_F chiral massless matter fields in the fundamental representation of the non-Abelian gauge group can be geometrically engineered by bringing N_f 3-branes near the 7-branes, i.e close to S [90]. The Higgs branches of these gauge theories should then be identified with the moduli spaces of the gauge instantons on S .

¹⁷The Abelian vector fields which arise in the $N = 2$ situation from the T^2 compactification from six to four dimensions do not appear in the modded $N = 1$ spectrum – see the discussion in the next chapter.

In our class of models, the space S is just given by the dP_9 surface which is the base of the $K3$ fibration of X_n^4 . The singularities of the elliptic four-fold fibrations are given by the singularities of the Weierstrass curve for \mathcal{B}_n^3 , given in eq.(6.23). So the non-Abelian gauge groups arise at the degeneration loci of eq.(6.23). However, with this observation we are in the same situation as in the $N = 2$ parent models, since the singularities of the modded elliptic curve \mathcal{B}_n^3 precisely agree with the singularities of the elliptic 3-folds X_n^3 in eq.(6.15). In other words, the non-Abelian gauge groups in the $N = 2$ and $N = 1$ models are identical. This observation can be explained from the fact that the gauge group enhancement already occurs in eight dimensions at the degeneration loci of the elliptic $K3$ surfaces as F-theory backgrounds. However, the underlying eight-dimensional $K3$ singularities are not affected by the \mathbf{Z}_2 operation on the coordinate z_1 , but are the same in eqs.(6.15) and (6.23).

In the following section about the heterotic dual models, we will explicitly determine the non-Abelian gauge groups and the possible Higgsing chains. We will show that after maximal Higgsing of the gauge groups the dimensions of the instanton moduli spaces, being identical with the dimensions of the Higgsing moduli spaces, precisely agree with $2h^{(2,1)}(X_n^4) - \rho + 2$ on the F -theory side; in addition we also verify that the ranks of the unbroken gauge groups after the complete Higgsing precisely match the ranks of the F-theory gauge groups, as given by $r(V)$ in table eq.(6.36).

6.4 Heterotic String on the $CY^{19,19}$

6.4.1 The $N = 2$ models: the heterotic string on $K3(\times T^2)$

In this section, we will construct the heterotic string compactifications dual to X_n^4 with $N = 1$ supersymmetry by \mathbf{Z}_2 modding of $N = 2$ heterotic string compactifications which are the duals of the F-theory models on $X_n^3 \times T^2$. Heterotic string models with $N = 2$ supersymmetry in four dimensions are obtained by compactification on $K3 \times T^2$ plus the specification of an $E_8 \times E_8$ gauge bundle over $K3$. So in the heterotic context, we have to specify how the \mathbf{Z}_2 modding acts both on the compactification space $K3 \times T^2$ as well as on the heterotic gauge bundle.

As above, the $N = 2$ heterotic models, that are dual to the F-theory compactifications on $X_n^3 \times T^2$, are characterized by turning on $(n_1, n_2) = (12 + n, 12 - n)$ ($n \geq 0$) instantons of the heterotic gauge group $E_8^{(I)} \times E_8^{(II)}$. Recall further, while the first $E_8^{(I)}$ is generically completely broken by the gauge instantons, the second $E_8^{(II)}$ is only completely broken for the cases $n = 0, 1, 2$; for bigger values of n there is a terminating gauge group $G^{(II)}$ of rank $r(V)$ which cannot be broken further. The quaternionic dimensions of the instanton moduli space of n instantons of a gauge group H , living on $K3$, was given by $\dim_Q(\mathcal{M}_{\text{inst}}(H, k)) = c_2(H)n - \dim H$, where $c_2(H)$ is the dual Coxeter number of H . Then, in the examples we are discussion, we derive the following formula for the quaternionic dimension of the instanton moduli space:

$$\dim_Q \mathcal{M}_{\text{inst}}(E_8^{(I)} \times H^{(II)}, n) = 112 + 30n + (12 - n)c_2(H^{(II)}) - \dim H^{(II)}; \quad (6.37)$$

here (for $n \neq 12$) $H^{(II)}$ is the commutant of the unbroken gauge group $G^{(II)}$ in $E_8^{(II)}$. Specifically, the following gauge groups $G^{(II)}$ and dimensions of instanton moduli spaces are derived:

n	$G^{(II)}$	$\dim_Q \mathcal{M}_{\text{inst}}$
0	1	224
2	1	224
4	$SO(8)$	252
6	E_6	302
12	E_8	472

(6.38)

The number of massless gauge singlet hypermultiplets is then simply given by

$$n_H = 20 + \dim_Q \mathcal{M}_{\text{inst}}, \quad (6.39)$$

where 20 corresponds to the complex deformations of $K3$. One finds perfect agreement, if one compares the spectra of F-theory on the 3-folds X_n^3 (see eqs.(6.20,6.21) and table (6.17)) with the spectra of the heterotic string on $K3$ with instanton numbers $(12+n, 12-n)$ (see eq.(6.39) and table (6.38)).

Note: On the heterotic side there is an perturbative gauge symmetry enhancement at special loci in the hypermultiplet moduli spaces. Specifically, by embedding the $SU(2)$ holonomy group of $K3$, namely the $SU(2)$ bundles with instanton numbers $(12+n, 12-n)$ in $E_8^{(I)} \times E_8^{(II)}$, the six-dimensional gauge group is broken to $E_7^{(I)} \times E_7^{(II)}$ (or $E_7^{(I)} \times E_8^{(II)}$ for $n = 12$); in addition one gets charged hyper multiplet fields, which can be used to Higgs the gauge group via several intermediate gauge groups down to the terminating groups. The dimensions of the Higgs moduli space, i.e. the number of gauge neutral hypermultiplets, agrees with the dimensions of the instanton moduli spaces eq.(6.37).

6.4.2 The $N = 1$ models: the heterotic string on $Z = (K3 \times T^2)/\mathbf{Z}_2$

Now let us construct the four-dimensional heterotic compactifications with $N = 1$ supersymmetry, which are dual to F-theory on X_n^4 , by \mathbf{Z}_2 modding of the heterotic string compactifications on $K3 \times T^2$. In the first step we discuss the \mathbf{Z}_2 modding of the compactification space $K3 \times T^2$ which results in a particular Calabi-Yau 3-fold Z :¹⁸

$$Z = \frac{(K3 \times T^2)}{\mathbf{Z}_2}. \quad (6.40)$$

Specifically, the \mathbf{Z}_2 -modding reduces $K3$ to the del Pezzo surface dP_9 . This corresponds to having on $K3$ a Nikulin involution of type $(10,8,0)$ with two fixed elliptic fibers in the $K3$ leading to

$$\begin{array}{ccc} K3 & \rightarrow & dP_9 \\ \downarrow & & \downarrow \\ \mathbf{P}_y^1 & \rightarrow & \mathbf{P}_{\tilde{y}}^1 \end{array} \quad (6.41)$$

induced from the quadratic base map $y \rightarrow \tilde{y} := y^2$ with the two branch points 0 and ∞ (being the identity along the fibers). In the Weierstrass representation of $K3$

$$K3 : \quad y^2 = x^3 - f_8(z)x - g_{12}(z), \quad (6.42)$$

¹⁸At the orbifold point of $K3$ one can construct Z as $T^6/(\mathbf{Z}_2 \times \mathbf{Z}_2)$, where one of the \mathbf{Z}_2 's acts freely, see e.g. [30].

the mentioned quadratic redefinition translates to the representation

$$dP_9 : \quad y^2 = x^3 - f_4(z)x - g_6(z) \quad (6.43)$$

of dP_9 (showing again the $8 = 5 + 7 - 3 - 1$ deformations). So the \mathbf{Z}_2 reduction of $K3$ to the non Ricci-flat dP_9 corresponds to the reduction of the X_n^3 to the non Calabi-Yau space \mathcal{B}_n^3 (cf. eqs.(6.15 and 6.23)). Representing $K3$ as a complete intersection in the product of projective spaces as $K3 = \left[\begin{array}{c} \mathbf{P}^2 \\ \mathbf{P}^1 \end{array} \middle| \begin{array}{c} 3 \\ 2 \end{array} \right]$, the \mathbf{Z}_2 modding reduces the degree in the \mathbf{P}^1 variable by half; hence the dP_9 can be represented as $dP_9 = \left[\begin{array}{c} \mathbf{P}_x^2 \\ \mathbf{P}_y^1 \end{array} \middle| \begin{array}{c} 3 \\ 1 \end{array} \right]$. This makes visible on the one hand its elliptic fibration over \mathbf{P}^1 via the projection onto the second factor; on the other hand the defining equation $C(x_0, x_1, x_2)y_0 + C'(x_0, x_1, x_2)y_1 = 0$ shows that the projection onto the first factor exhibits dP_9 as being a \mathbf{P}_x^2 blown up in 9 points (of $C \cap C'$), thus having as nontrivial hodge number (besides b_0, b_4) only $h^{1,1} = 1 + 9$. Furthermore, the dP_9 has 8 complex structure moduli: they can be seen as the parameter input in the construction of blowing up the plane in the 9 intersection points of two cubics (the ninth of which is then always already determined as they sum up to zero in the addition law on the elliptic curve; so one ends up with $8 \times 2 - 8$ parameters).

As in the dual F-theory description a second dP_9 emerges by fibering the T^2 in eq.(6.40) over the \mathbf{P}^1 base of dP_9 . So in analogy to eq.(6.27) the heterotic Calabi-Yau 3-fold X_{het}^3 , which is elliptically fibered over dP_9 , has the following fiber product structure

$$Z = dP_9 \times_{\mathbf{P}^1} dP_9. \quad (6.44)$$

The number of Kähler deformations of Z is given by the sum of the deformations of the two dP_9 's minus one of the common \mathbf{P}^1 base, i.e. $h^{(1,1)}(Z) = 19$. Similarly we obtain $h^{(2,1)}(Z) = 8 + 8 + 3 = 19$. This Calabi-Yau 3-fold is in fact well known, being one of the Voisin-Borcea Calabi-Yau spaces. It can be obtained from $K3 \times T^2$ by the Voisin-Borcea involution, which consists in the ‘del Pezzo’ involution (type (10,8,0) in Nikulins classification) with two fixed elliptic fibers in the $K3$ combined with the usual “-”-involution with four fixed points in the T^2 . Writing $K3 \times T^2$ as $K3 \times T^2 = \left[\begin{array}{c} \mathbf{P}^2 \\ \mathbf{P}^1 \\ \mathbf{P}^2 \end{array} \middle| \begin{array}{cc} 3 & 0 \\ 0 & 2 \\ 0 & 3 \end{array} \right]$ the Voisin-Borcea involution changes this to $Z = \left[\begin{array}{c} \mathbf{P}^2 \\ \mathbf{P}^1 \\ \mathbf{P}^2 \end{array} \middle| \begin{array}{cc} 3 & 0 \\ 1 & 1 \\ 0 & 3 \end{array} \right] = dP \times_{\mathbf{P}^1} dP$. Observe that the base of the elliptic fibration of Z is given by the dP_9 surface which ‘emerges’ (from the trivial elliptic fibration) after the \mathbf{Z}_2 modding.

After having described the \mathbf{Z}_2 modding of $K3 \times T^2$, we will now discuss how this operation acts on the heterotic gauge bundle. Recall that in the $N = 2$ heterotic models on $K3 \times T^2$ the heterotic gauge group $E_8^{(I)} \times E_8^{(II)}$ lives on the four-dimensional space $K3$. We will now consider a $N = 1$ situation where, after the \mathbf{Z}_2 modding, the heterotic gauge group still lives on a four manifold, namely on the del Pezzo surface dP_9 , which arises from the \mathbf{Z}_2 modding of the $K3$ surface. Then the complex dimension of the instanton moduli space of k gauge instantons of a gauge group H , which lives on dP , is given by

$$\dim_C \mathcal{M}_{\text{inst}}(H, k) = 2c_2(H)k - \dim H. \quad (6.45)$$

The action of the \mathbf{Z}_2 modding on the gauge bundle is now defined in such a way that the gauge instanton numbers are reduced by half (think of the limit case of pointlike instantons):

$$k_{1,2} = \frac{n_{1,2}}{2}. \quad (6.46)$$

So the total number of gauge instantons in $E_8^{(I)} \times E_8^{(II)}$ will be reduced by two, i.e. $k_1 + k_2 = 12$ and we consider $(k_1, k_2) = (6 + \frac{n}{2}, 6 - \frac{n}{2})$ instantons in $E_8^{(I)} \times E_8^{(II)}$. The reduction of the total instanton number by half from 24 to 12 can be explained from the observation that on the F-theory side the tad-pole anomaly can canceled either by $\chi/24 = 12$ 3-branes or by 12 gauge instantons of the gauge group H , which lives over the four manifold $S = dP_9$. So, with $k_1 + k_2 = 12$ and using eq.(6.45), we can compute the complex dimensions of the instanton moduli space for the gauge group $E_8^{(I)} \times H^{(II)}$, where again $H^{(II)}$ is the commutant of the gauge group $G^{(II)}$ which cannot be further broken by the instantons:

$$\dim_C \mathcal{M}_{\text{inst}}(E_8^{(I)} \times H^{(II)}, n) = 112 + 30n + (12 - n)c_2(H^{(II)}) - \dim H^{(II)}. \quad (6.47)$$

This result precisely agrees with the quaternionic dimensions of the instanton moduli space, eq.(6.37), in the unmodded $N = 2$ models. So we see that we obtain as gauge bundle deformation parameters of the heterotic string on Z the same number of massless, gauge neutral $N = 1$ *chiral* multiplets as the number of massless $N = 2$ *hyper* multiplets of the heterotic string on $K3$. This means that the \mathbf{Z}_2 modding keeps just one of the two chiral fields in each $N = 2$ hyper multiplet in the massless sector. These chiral multiplets describe the Higgs phase of the $N = 1$ heterotic string compactifications.

The gauge fields in $N = 1$ heterotic string compactifications on Z are just given by those gauge fields which arise from the compactification of the heterotic string on $K3$ to six dimensions; therefore they are invariant under the \mathbf{Z}_2 modding. However, the complex scalar fields of the corresponding $N = 2$ vector multiplets in four dimensions do not survive the \mathbf{Z}_2 modding. Therefore, there is no Coulomb phase in the $N = 1$ models in contrast to the $N = 2$ parent compactifications. Also observe that the two vector fields, commonly denoted by T and U , which arise from the compactification from six to four dimensions on T^2 , disappear from the massless spectrum after the modding. This is expected since the Calabi-Yau space has no isometries which can lead to massless gauge bosons. Finally, the $N = 2$ dilaton vector multiplet S is reduced to a chiral multiplet in the $N = 1$ context.

These relations between the spectra of the $N = 1$ and $N = 2$ models can be understood from the observation that the considered \mathbf{Z}_2 modding corresponds to a spontaneous breaking of $N = 2$ to $N = 1$ spacetime supersymmetry [91].

In summary, turning on $(6 + \frac{n}{2}, 6 - \frac{n}{2})$ gauge instantons of $E_8^{(I)} \times E_8^{(II)}$ in our class of $N = 1$ heterotic string compactifications on Z , the unbroken gauge groups G^{II} as well as the number of remaining massless chiral fields (not counting the geometric moduli from Z , see next paragraph) agree with the unbroken gauge groups and the number of massless hyper multiplets (again without the 20 moduli from $K3$) in the heterotic models on $K3$ with $(12 + n, 12 - n)$ gauge instantons. The specific gauge groups and the numbers of chiral fields are already summarized in table (6.38).

Now comparing with the F-theory spectra, we first observe that the ranks of the gauge groups after maximally possible Higgsing perfectly match in the two dual descriptions (see tables (6.36) and (6.38)).

Next compare the number of chiral $N = 1$ moduli fields in the heterotic/F-theory dual pairs. First, looking at the Hodge numbers of the dual F-theory fourfolds X_n^4 , as given in table (6.36) we recognize that

$$2h^{(2,1)}(X_n^4) - (\rho - 2) = \dim_C \mathcal{M}_{\text{inst}}. \quad (6.48)$$

Let us argue this independent of the case by case calculation. Namely, using $C_F = 38 + (2h^{(2,1)}(X_n^4) - (\rho - 2)) = n_H + 18$ (cfr. eqs.(6.34) and (6.35)) and $n_H = 20 + \dim_Q \mathcal{M}_{\text{inst}} = 20 + \dim_C \mathcal{M}_{\text{inst}}$, one gets

$$C_F = 38 + (2h^{(2,1)}(X_n^4) - (\rho - 2)) = 38 + \dim_C \mathcal{M}_{\text{inst}} \quad (6.49)$$

On the heterotic side, the total number C_{het} of chiral moduli fields is given by the dimension of the gauge instanton moduli space plus the number of geometrical moduli

$h^{(1,1)}(Z) + h^{(2,1)}(Z)$ from the underlying Calabi-Yau space Z , which is 38 for our class of models, i.e.

$$C_{het} = 38 + \dim_C \mathcal{M}_{inst}. \quad (6.50)$$

Discussion: So we have shown that the massless spectra of Abelian vector multiplets and of the gauge singlet chiral plus antichiral fields agree for all considered dual pairs. The next step in the verification of the $N = 1$ string-string duality after the comparison of the massless states is to show that the interactions, i.e. the $N = 1$ effective action, agree. In particular, the construction of the superpotentials is important to find out the ground states of these theories. This was already done [28, 30] for one particular model (the model with $k = 9$, $n = 0$), where on the heterotic, side the superpotential was entirely generated by world sheet instantons. It would be interesting to see whether space time instantons would also contribute to the heterotic superpotential in some other models and whether supersymmetry can be broken by the superpotential. In addition, it would also be interesting to compare the holomorphic gauge kinetic functions in $N = 1$ dual string pairs, in particular in those models obtained from $N = 2$ dual pairs by \mathbf{Z}_2 moddings respectively by spontaneous supersymmetry breaking from $N = 2$ to $N = 1$.

6.4.3 Non-Abelian gauge groups and Higgsing chains

For the computation of C and $r(V)$ we have considered a generic point in the moduli space where the gauge group is broken as far as possible to the group G^{II} by the vacuum expectation values of the chiral fields. In this section, we now want to determine the non-Abelian gauge groups plus their matter content which arise in special loci of the moduli space. Since the $N = 1$ gauge bundle is identical to the $N = 2$ bundle, which is given by $SU(2) \times SU(2)$, the maximally unbroken gauge group is for $k_1, k_2 \geq 3$ (i.e. $n \leq 6$) given by $E_7^{(I)} \times E_7^{(II)}$; the $N = 1$ chiral representations follow immediately from the $N = 2$ hypermultiplet representations and transform as

$$E_7 \times E_7 : \quad (k_1 - 2)(\mathbf{56}, \mathbf{1}) + (k_2 - 2)(\mathbf{1}, \mathbf{56}) + (4(k_1 + k_2) - 6)(\mathbf{1}, \mathbf{1}). \quad (6.51)$$

Higgsing $E_7^{(I)} \times E_7^{(II)}$ to $E_6^{(I)} \times E_6^{(II)}$ one is left with chiral matter fields in the following representations of the gauge group $E_6^{(I)} \times E_6^{(II)}$:

$$E_6 \times E_6 : \quad (k_1 - 3)[(\mathbf{27}, \mathbf{1}) + (\overline{\mathbf{27}}, \mathbf{1})] + (k_2 - 3)[(\mathbf{1}, \mathbf{27}) + (\mathbf{1}, \overline{\mathbf{27}})]$$

$$+(6(k_1 + k_2) - 16)(\mathbf{1}, \mathbf{1}). \quad (6.52)$$

When $k_1 = 12$ ($n = 12$) the gauge group is $E_6^{(I)} \times E_8^{(II)}$ with chiral matter fields

$$E_6 \times E_8 : \quad 9[(\mathbf{27}, \mathbf{1}) + (\overline{\mathbf{27}}, \mathbf{1})] + 64(\mathbf{1}, \mathbf{1}). \quad (6.53)$$

Since $k_1 \geq 6$, the number of $(\mathbf{27}, \mathbf{1}) + (\overline{\mathbf{27}}, \mathbf{1})$ is always big enough that the first $E_6^{(I)}$ can be completely broken. On the other hand, only for the cases $n = 0, 2$ the group $E_6^{(II)}$ can be completely Higgsed away by giving vacuum expectation values to the fields $(\mathbf{1}, \mathbf{27}) + (\mathbf{1}, \overline{\mathbf{27}})$. For $k_2 = 4$ ($n = 4$), $E_6^{(II)}$ can be only Higgsed to the group $G^{(II)} = SO(8)$, and for $k_2 = 3$ ($n = 6$) there are no charged fields with respect to $E_6^{(II)}$ such that the terminating gauge group is just $E_6^{(II)}$. Clearly, the ranks of these gauge groups are in agreement with the previous discussions, i.e. with the results for $r(V)$; in addition, assuming maximally possible Higgsing of both gauge group factors the complex dimension of the Higgs moduli space agrees with the dimensions of the instanton moduli spaces as given in eq.(6.47) and in table (6.38).

Consider the Higgsing of, say, the first gauge group $E_6^{(I)}$. Namely, like in the $N = 2$ cases [89], [92], it can be Higgsed through the following chain of Non-Abelian gauge groups:

$$E_6 \rightarrow SO(10) \rightarrow SU(5) \rightarrow SU(4) \rightarrow SU(3) \rightarrow SU(2) \rightarrow SU(1). \quad (6.54)$$

In the following we list the spectra for all gauge groups within this chain:

$$SO(10) : \quad (k_1 - 3)(\mathbf{10} + \overline{\mathbf{10}}) + (k_1 - 4)(\mathbf{16} + \overline{\mathbf{16}}) + (8k_1 - 15)\mathbf{1}, \quad (6.55)$$

$$SU(5) : \quad (3k_1 - 10)(\mathbf{5} + \overline{\mathbf{5}}) + (k_1 - 5)(\mathbf{10} + \overline{\mathbf{10}}) + (10k_1 - 24)\mathbf{1}, \quad (6.56)$$

$$SU(4) : \quad (4k_1 - 16)(\mathbf{4} + \overline{\mathbf{4}}) + (k_1 - 5)(\mathbf{6} + \overline{\mathbf{6}}) + (16k_1 - 45)\mathbf{1}, \quad (6.57)$$

$$SU(3) : \quad (6k_1 - 27)(\mathbf{3} + \overline{\mathbf{3}}) + (24k_1 - 78)\mathbf{1}, \quad (6.58)$$

$$SU(2) : \quad (12k_1 - 56)\mathbf{2} + (36k_1 - 133)\mathbf{1}, \quad (6.59)$$

$$SU(1) : \quad (60k_1 - 248)\mathbf{1}. \quad (6.60)$$

In order to keep contact with our previous discussion, we see that the number of massless chiral fields at a generic point in the moduli space, i.e. for complete Higgsing down to $SU(1)$, is given by $60k_1 - 248 + 2k_2 c_2(H^{(II)}) - \dim H^{(II)}$ which precisely agrees with $\dim_C \mathcal{M}_{\text{inst}}$

6.5 Standard embedding and Higgsing

In this final section, let us consider the standard embedding [51]! We will do this for two reasons: first, the standard embedding involves the anomaly cancellation without fivebranes ($n_5 = 0$). Thus, under duality we should expect that the number of threebranes vanishes also. Then assuming no four-flux is turned on and also no instantons, inside the compact part of the worldvolume of the seven-brane, are tuned on, we should expect $\chi(X_4) = 0$; second, we can easily determine the number of matter multiplets transforming as **27**'s of E_6 , since left handed **27**'s come from elements of $H^1(Z, TZ)$, while right handed $\overline{\mathbf{27}}$'s come from elements of $H^1(Z, TZ^*)$ and the net amount of chiral matter is then given by (c.f. section 2) $N_{gen} = \frac{1}{2}c_3(TZ)$. In addition we can determine the number of bundle moduli m_{bun} , and we can try to understand the analog of the 6D complete Higgsing process!

Now, take the tangent bundle TZ of Z which has $SU(3)$ holonomy and identify it with the gauge field that belongs to the subgroup of E_8 (which commutes with E_6) and thus breaking E_8 to E_6 . So one ends up with the gauge group $E_8 \times E_6$ where we think of the E_8 as being the "hidden sector" and the E_6 as being the "observable sector" which leads to massless charged matter. The adjoint of E_8 decomposes under $E_6 \times SU(3)$ as

$$\mathbf{248} = (\mathbf{78}, \mathbf{1}) + (\mathbf{27}, \mathbf{3}) + (\overline{\mathbf{27}}, \overline{\mathbf{3}}) + (\mathbf{1}, \mathbf{8}). \quad (6.61)$$

The compactification is specified by $h^{1,1}(Z)$, $h^{2,1}(Z)$ and $h^1(Z, \text{End}(TZ))$ where TZ denotes the tangent bundle to Z . Further, let us assume that Z can be represented by its smooth Weierstrass model. So we have (as above) $h^{1,1}(Z) = 11 - c_1^2$ and $h^{2,1}(Z) = 11 + 29c_1^2$. We computed the index of TZ in section 3.3.2, which we will use now, recall that the index is given by $I = -46 - 28c_1^2$ and thus leading to the number of bundle moduli

$$m_{bun} = -46 - 28c_1^2 + 2n_o \quad (6.62)$$

Since $m_{bun} > 0$ we get a condition for the bundle moduli which are odd under the τ -involution of Z : $n_o \geq 46 + 28c_1^2$.

3-branes: Recall, the number of threebranes can be written as (c.f. section 5)

$$\frac{\chi(X_4)}{24} = 2 + \frac{1}{4}(40 + 28c_1^2 + I - 16 + r(V)) = n_3 \quad (6.63)$$

where $r(V)$ denotes the rank of the unbroken 4D gauge group. In our case we find with $I = -46 - 28c_1^2$ and $r(V) = 14$

$$n_3 = 0. \quad (6.64)$$

Higgsing: In the standard embedding, the number of $\mathbf{27}$'s resp. $\overline{\mathbf{27}}$'s is given by $h^{1,1}(Z)$ resp. $h^{2,1}(Z)$, so we expect a total number of $22 + 28c_1^2$ charged multiplets. Let us now Higgs the E_6 completely, we get

$$C_{Higgs} = (22 + 28c_1^2)27 - 78 = 516 + 756c_1^2. \quad (6.65)$$

We are left with an unbroken E_8 and a number C_{Higgs} of $N = 1$ neutral chiral multiplets. Also taking into account the geometrical moduli $(h^{1,1}, h^{2,1})$ and the bundle moduli m_{bun} we end up with

$$C_{het} = 492 + 756c_1^2 + 2n_o \quad (6.66)$$

further, we have the net generation number $N_{gen} = \frac{1}{2}c_3(V) = 30c_1^2$.

Let us try to shed some light on C_{het} !

A striking fact is the appearance of 492, indicating a possible relation to the six-dimensional heterotic $(n_1, n_2) = (0, 24)$ compactification which leads to an unbroken $E_8 \times E_7$ gauge group (the standard embedding mentioned in section 5.2.1). In particular, complete Higgsing leads to $n_V = 248$, $n_T = 1$ and $n_H = 427 + 45 + 20 = 492$. Now, following the 'spirit' of the last section, we should compactify the 6D theory on T^2 leaving us with $N = 2$ supersymmetry in 4D. To obtain a $N = 1$ theory we have to mod out by a group G . To determine G let us make a short digression! If we recall Z , we get a hint from the Hodge numbers of Z for a viable G . We find that for $c_1(B_2) = 0$, we obtain the well known Voisin-Borcea model (11,11), which can be obtained from $K3 \times T^2$ by the Nikulin involution of type (10,10,0) which has no fixed fibers in the $K3$, so we have $Z = \frac{K3 \times T^2}{(\sigma, -1)}$ where '-1' is a involution on T^2 with four fixed points. Moreover, in this case we can think of Z as being elliptically fibered over the Enriques surface $K3/\sigma$. So, in the following let us restrict ourselves to the (11,11) model! Thus C_{het} reduces to $C_{het} = 492 + 2n_o$. Let us return to our model on $K3 \times T^2$. Now recall, the 4D $N = 2$ spectrum is obtained from 6D, upon T^2 compactification, as follows: the 6D vectors become 4D vectors, the 6D tensor becomes a abelian $N = 2$ vector field, the hyper multiplets become 4D $N = 2$ hyper multiplets and in addition we get 2 vectors related to the T and U moduli of T^2 . Let us discuss the modding of $K3 \times T^2$. We will consider that, after modding, the heterotic gauge group still lives on a four manifold, which is the Enriques surface. The complex dimension of the instanton moduli space is given by $\dim_C \mathcal{M}_{inst} = 2nc_2(V) - n^2 + 1$ and if we assume

that the modding reduces the instanton numbers by half, we get $\dim_C \mathcal{M}_{inst} = 45$. So, we are left with the explanation of $2n_o$. Understanding $2n_o$ would then lead to an interesting picture, where the complete Higgsed 6D (0,24) model would correspond to the 4D $(h^{1,1}, h^{2,1} = 11, 11)$ model, where we consider again the complete Higgsed case of the 4D standard embedding. We will close with this neat observation, which leads us to summary and outlook!

Summary and Outlook

We have studied four-dimensional $N = 1$ het/F-theory duality. The duality involves the understanding of vector bundles on elliptic fibrations. Vector bundles can be constructed using the parabolic or spectral cover bundles construction. We adopted the parabolic approach and constructed a class of $SU(n)$ vector bundles which have n odd and a certain congruence relation ($\eta \equiv 0(\text{mod } n)$). In particular, these bundles have non-vanishing $c_3(V)$ (which is related to a non-zero net amount of chiral matter). Then we compared our results with the spectral cover approach and found agreement for a certain twist ($\lambda = \frac{1}{2n}$) of the line bundle on the spectral cover.

We also computed the number of bundle moduli of E_8 bundles, using a character valued index theorem, which was first used in its specific form by Friedman, Morgan and Witten for $SU(n)$ vector bundles.

Further, consistent F-theory compactification requires a number of space-time filling threebranes, which should turn, under duality, into heterotic fivebranes, which wrap the elliptic fiber. This matching has been established by Friedman, Morgan and Witten for E_8 bundle, leaving no unbroken gauge group. We could extend this matching to the general case of $SU(n)$ vector bundles, leaving an unbroken gauge group on the heterotic side.

A het/F-theory duality also involves the comparison between massless spectra. We found a complete matching of the spectra under two assumptions: first, no four-flux is turned on; second, no instanton background inside the compact part of the sevenbrane is turned on.

Then we considered $N = 1$ dual het/F-theory pairs. In particular, we constructed a chain of dual pairs by \mathbf{Z}_2 -modding of corresponding dual pairs with $N = 2$ supersymmetry, which were obtained from six-dimensional dual pairs after T^2 compactification.

Apart from the moduli matching, one should also understand the part of the 4D heterotic spectrum which corresponds to charged matter. We made a first step in this direction due to the computation of $c_3(V)$ in the parabolic approach; recently, this computation was also performed in the spectral cover approach [47]. However, in both approaches a difficulty appears. For τ -invariant vector bundles, we can apply an index theorem to compute the number of bundle moduli, but these bundles have $c_3(V) = 0$ and therefore lead to a zero net amount of chiral matter; in contrast, vector bundles with $c_3(V) \neq 0$ are not τ -invariant and thus we lose the control over the bundle moduli (since we cannot apply the character valued index theorem). This seems to be a rather

technical problem, but it has to be solved in order to understand possible couplings between gauge singlets and chiral matter fields.

Another problem which begs further analysis is to work out a refined het/F-theory matter dictionary, which requires an improved understanding of intersecting seven-branes [62].

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